Page 1

21 August 2020 09:33

Def. A set $E \subseteq X$ is called open if $\forall e \in E$, $\exists r > 0$ s.t. $B(e,r) \subseteq E$.

Def. A neighbourhood of a point $x \in E$ is a ball $B(x,r)$ for some $r > 0$.

Theo. An arbitrary union of open sets is open.

Proof. Let $\{E_\alpha\}_{\alpha \in A}$ be open.

Then $x \in \bigcup_{\alpha \in A} E_\alpha \Rightarrow \exists \alpha \in A$ s.t. $x \in E_{\alpha_0}$

$\Rightarrow \exists r > 0$ s.t. $B(x,r) \subseteq E_{\alpha_0}$

$\Rightarrow B(x,r) \subseteq \bigcup_{\alpha \in A} E_\alpha \Rightarrow \bigcup_{\alpha \in A} E_\alpha$ is open. $\square$

What about an arbitrary intersection? No!

Consider $\left\{\left(\frac{-1}{n}, \frac{1}{n}\right)\right\}_{n \in \mathbb{N}}$. The intersection of these sets is $\{0\}$, which is not open.

Ex. Prove that a finite intersection of open sets is open.

Theo. Let E be an open set in $\mathbb{R}$. Then E is an at most countable union of disjoint open intervals in $\mathbb{R}$.

Proof. For each $x \in E$, consider

$$J_x = \bigcup_{\substack{x \in I \\ I \text{ is an open interval}}} I.$$

J_x is the maximal open interval that contains x.

Now, for $x, y \in E$, either $J_x = J_y$ or $J_x \cap J_y = \emptyset$. That is, E is a union of disjoint open intervals. This implies that it is an at most countable union.

(As there is a rational number in each interval, the number of intervals is $\leq |\mathbb{Q}| = |\mathbb{N}|$). $\square$

Does an analogous result hold for $\mathbb{R}^2$?

Def. Let (X,d) be a metric space and $E \subseteq X$. A point $x \in X$ is called a limit point of E if for all $r > 0$,

$$B(x,r) \cap E \setminus \{x\} \neq \emptyset.$$

That is, there exist points in the set arbitrarily close to x. $\forall r > 0$, $\exists y \in E$ s.t. $y \neq x$, $d(x,y) < r$

Example. Consider $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subseteq \mathbb{R}$.

0 is a limit point of E. (Consequence of the Archimedean property).

Ex. Prove that 0 is the only limit point of the above set.

We denote the set of limit points of a set E by E'.

Ex. Prove that $\mathbb{Q}' = \mathbb{R}$.

Ex. Let $E \subseteq \mathbb{R}$ be a finite set. Prove that $E' = \emptyset$.

Def. Let (X,d) be a metric space and $E \subseteq X$. E is said to be closed if $E' \subseteq E$.

Note that $\mathbb{Q}$ is neither open nor closed (in $\mathbb{R}$)

Also, $\emptyset$ and $\mathbb{R}$ are both open and closed (in $\mathbb{R}$).

We expect some relation between open sets and closed sets.
(if anything, from the names "open" and "closed")

Theo. Let (X,d) be a metric space and $E \subseteq X$. Then
E is closed $\Leftrightarrow E^c$ is open.

Page 2

26 August 2020 09:34

Proof: Let E^c be closed. For $x \in E$, x is not a limit point of E^c.
$\Rightarrow \exists r>0$ s.t. $E^c \cap B(x,r) \setminus \{x\} = \emptyset$
$\Rightarrow E^c \cap B(x,r) = \emptyset$
$\Rightarrow B(x,r) \subseteq E \Rightarrow E$ is open.
Let E be open. Let x be a limit point of E^c s.t. $x \notin E^c$.
Then $\forall r>0$ $B(x,r) \cap E^c \setminus \{x\} \neq$
$\Rightarrow \forall r>0$ $B(x,r) \cap E^c \neq \emptyset$.
However, $\exists r_0>0$ $B(x,r_0) \subseteq E$ (as $x \in E$ and E is open)
Contradiction! $\Rightarrow E^c$ is closed. ■

Corollary. 1) Arbitrary intersection of closed sets is closed.
2) Finite union of closed sets is closed.
$\hookrightarrow$ Not arbitrary: example. $\bigcup_{n=1}^{\infty} [\frac{1}{n}, 1-\frac{1}{n}] = (0,1)$

Def. Let (X,d) be a metric space and $E \subseteq X$. Then the closure of the set E, denoted $\bar{E}$, is given by
$$\bar{E} = E \cup E'$$

For example, $\bar{\mathbb{Q}} = \mathbb{R}$.

Thm. Given any E,
1) $\bar{E}$ is closed.
2) $E = \bar{E} \iff E$ is closed.
3) $\bar{E}$ is the smallest closed set that contains E.
$\hookrightarrow$ the intersection of all closed sets that contain E.

Proof. 1) We shall show that $\bar{E}^c$ is open. Let $y \in \bar{E}^c \Rightarrow y$ is neither in E nor a limit point of E.

$\Rightarrow \exists r>0 \quad B(y,r) \cap E \setminus \{y\} = \emptyset$

$\Rightarrow \quad B(y,r) \cap E = \emptyset \quad$ Also, $B(y,r) \cap E' = \phi$

$\Rightarrow \quad B(y,r) \cap \overline{E} = \emptyset \quad \Rightarrow B(y,r) \subseteq \overline{E}^c$

$\Rightarrow \overline{E}$ is closed.

2) $\overline{E} = E \cup E'$

$E = \overline{E} \Leftrightarrow E' \subseteq E \Leftrightarrow E$ is closed.

3) let $F \supseteq E$ be a closed set. This implies that $E' \subseteq F' \subseteq F$.

As $E \subseteq F$ and $E' \subseteq F$, $\overline{E} \subseteq F$.

$\Rightarrow \overline{E}$ is the smallest closed set that contains E. $\blacksquare$

Def. Let (X,d) be a metric space. E is said to be **dense (in X)** if $\overline{E} = X$.

Ex. Prove that E is dense in X iff for any open $U \subseteq X$, $E \cap U = \emptyset$.

For example, $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.

Def. Let (X,d) be a metric space and $E \subseteq X$. An **open cover (of E)** is a collection of open sets $\{G_\alpha\}_{\alpha \in A}$ such that

$$E \subseteq \bigcup_{\alpha \in A} G_\alpha$$

Def. Let (X,d) be a metric space and $E \subseteq X$. E is said to be **compact** if every open cover of E has a finite subcover.

i.e. if $E \subseteq \bigcup_{\alpha \in A} G_\alpha$ where each G_α is open, then $\exists \alpha_1, \alpha_2, \cdots, \alpha_n \in A$ s.t. $E \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$.

Example. Any finite set is a compact set.

(We can choose sets $G_1, G_2, \cdots, G_n$ such that $x_i \in G_i$ where the finite set is $\{x_1, x_2, x_3, \cdots, x_n\}$)

Page 4

26 August 2020 10:18

$\mathbb{R}$ is not compact.
(Consider the open cover $\{(n-1, n+1)\}_{n\in\mathbb{Z}}$)

$(0,1)$ is not compact.
(Consider the open cover $\{(\frac{1}{n}, 2)\}_{n\in\mathbb{N}}$)

Ex. Let $E \subseteq \mathbb{R}$ be not bounded. Can E be compact?
(Consider the open cover of $\mathbb{R}$ we used earlier)

$\Rightarrow$ Any compact set is bounded.

Theo. Let (X,d) be a metric space and $E \subseteq X$. If E is compact, then E is closed.

Proof. We shall equivalently prove that E^c is open.
Let $p \in E^c$ and $q \in E$. Then $d(p,q) > 0$.
For each such q, choose r_q s.t. $0 < r_q < \frac{d(p,q)}{2}$.

Consider $\{B(q, r_q)\}_{q\in E}$. This is clearly an open cover of E.

As E is compact, there is a finite subcover:

$$E \subseteq B(q_1, r_{q_1}) \cup B(q_2, r_{q_2}) \cup \cdots \cup B(q_n, r_{q_n})$$

for some $q_1, q_2, \ldots, q_n \in E$.

Let $r = \min\{r_{q_1}, r_{q_2}, \ldots, r_{q_n}\}$.
We now claim that $E \cap B(p,r) = \emptyset$
Let us assume otherwise. If $x \in E \cap B(p,r)$.
Then $d(x,p) < r$ and also,
$x \in E \subseteq \bigcup_{1\le i\le n} B(q_i, r_{q_i}) \Rightarrow x \in B(q_j, r_{q_j})$ for some j

$$\Rightarrow d(p,q_j) \le d(x,p) + d(x,q_j) < r + r_{q_j} \le 2r_{q_j} < d(p,q_j)$$

Contradiction! As $B(p,r) \subseteq E^c$, E is closed. ■

Page 5

26 August 2020 10:47

Ex. Prove that $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is compact.

We now go towards the characterization of compact sets in $\mathbb{R}^n$.

Lemma. Let (X,d) be a metric space and $E \subseteq X$ be compact. If $F \subseteq E$ is closed, then F is compact.

Proof. Let $(V_\alpha)_{\alpha \in A}$ be an open cover of F. Then $(V_\alpha)_{\alpha \in A} \cup \underbrace{(X \setminus F)}_{\text{open set}}$ is an open cover of E. $\exists\, \alpha_1, \alpha_2, \dots, \alpha_n \in A$ such that

$$F \subseteq E \subseteq V_{\alpha_1} \cup \dots \cup V_{\alpha_n} \cup (X \setminus F)$$
$$\Rightarrow F \subseteq V_{\alpha_1} \cup V_{\alpha_2} \cup \dots \cup V_{\alpha_n}$$
$$\Rightarrow F \text{ is compact.} \quad \blacksquare$$

Lemma. Let $(I_n)_{n \in \mathbb{N}}$ be a sequence of closed and bounded intervals in $\mathbb{R}$ such that $I_1 \supseteq I_2 \supseteq \cdots$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Let $I_n = [a_n, b_n]$ for each $n \in \mathbb{N}$.

Let $E = \{a_n : n \in \mathbb{N}\}$. $E \neq \emptyset$ and E is bounded above by b_1.

Let $x = \sup E$.

We claim that $x \in \bigcap_{n=1}^{\infty} I_n$.

For $n, m \in \mathbb{N}$, $\quad a_n \leq a_{n+m} \leq b_{n+m} \leq b_n$

$$\Rightarrow x \leq b_m \ \forall m. \text{ We also have } a_m \leq x \ \forall m.$$
$$\Rightarrow x \in [a_m, b_m] \ \forall m$$
$$\Rightarrow x \in \bigcap_{n=1}^{\infty} [a_n, b_n]. \quad \blacksquare$$

(Where did we use closedness and boundedness?)

$\downarrow$

$\{(1, \frac{1}{n}) : n \in \mathbb{N}\}$ and $\{[n, \infty) : n \in \mathbb{N}\}$

Theo. [Characterization of compact sets in $\mathbb{R}^n$] Let $E \subseteq \mathbb{R}^n$. E is compact if and only if E is closed and bounded.

Proof. Claim 1. Any closed and bounded interval is compact.

Suppose $(G_\alpha)_{\alpha \in A}$ is an open cover of $\underbrace{[a,b]}_{= I_1}$ that does not have a finite subcover.

Then at least one of $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ does not have a finite subcover, let it be I_2. We can similarly split I_2 to get I_3. Continue this process to get a sequence of closed and bounded intervals $(I_n)_{n \in \mathbb{N}}$ s.t. $I_1 \supseteq I_2 \supseteq \cdots$ and none of them has a finite subcover of $(G_\alpha)_{\alpha \in A}$.

Further, if $x, y \in I_n$, then $d(x,y) \leq \frac{b-a}{2^{n-1}}$.

Let $x \in \bigcap_{n=1}^{\infty} I_n \subseteq \bigcup_{\alpha \in A} G_\alpha \Rightarrow x \in \underbrace{G_{\alpha_0}}_{\text{open set}}$ for some $\alpha_0 \in A$.

$\Rightarrow \exists r > 0 \quad B(x,r) \subseteq G_{\alpha_0}$

There then exists n_0 s.t. $\frac{b-a}{2^{n_0-1}} < r \Rightarrow I_{n_0} \subseteq B(x,r) \subseteq \underbrace{G_{\alpha_0}}_{\text{finite subcover}}$

Contradiction! $\Rightarrow$ Any closed and bounded interval is compact.

Claim 2. Any closed and bounded set in $\mathbb{R}$ is compact.

Let E be a closed and bounded set and $I \supseteq E$ be a closed bounded interval. Then as I is compact, $\underbrace{E}_{\text{a closed subset of } I}$ is compact.

We have already proved that a compact subset of $\mathbb{R}$ is closed and bounded. This completes the proof.

(The proof is similar for $\mathbb{R}^n$) ■

Let us now consider what sets have limit points.
Clearly not every set (finite sets, $\mathbb{N}$, etc.)

Theo. Let (X,d) be a metric space and $E \subseteq X$ be compact. Then any infinite subset of E has a limit point in E.

Proof. Suppose no point of E is a limit point of an infinite subset $F \subseteq E$.
$\Rightarrow$ For each $x \in E$, $\exists r_x > 0$ s.t. $B(x, r_x) \cap F \setminus \{x\} = \emptyset$.
$\Rightarrow$ $B(x, r_x)$ contains at most one point from F.
Note that $\bigcup_{x \in E} B(x, r_x)$ is an open cover of E, a compact set.

Let $F \subseteq E \subseteq \bigcup_{i=1}^{n} B(x_i, r_{x_i})$ $\Rightarrow$ F contains at most n points.

Contradiction! (F is an infinite set)
$\Rightarrow$ Some point of E is a limit point of F.

Corollary. Every infinite bounded set in $\mathbb{R}$ has a limit point in $\mathbb{R}$.
(Bolzano-Weierstrass Theorem)

(The set is a subset of some closed and bounded interval)

The converse of the above theorem holds as well.

Theo. Let (X,d) be a metric space and $E \subseteq X$ s.t any infinite subset of E has a limit point in E. Then E is compact.

Proof. We shall prove the result for $E \subseteq \mathbb{R}$.
Suppose E is not bounded. For each $n \in \mathbb{N}$, let $x_n \in E$ s.t. $|x_n| > n$.
Then $\{x_1, x_2, \ldots\} \subseteq E$ has no limit point in E. $\Rightarrow$ E is bounded.
Suppose E is not closed. Then $\exists x_0 \in E^c$ which is a limit point of E.

$\Rightarrow \forall n \in \mathbb{N} \quad E \cap B(x_0, \frac{1}{n}) \setminus \{x_0\} \neq \emptyset$.

Let $x_n \in E \cap B(x_0, \frac{1}{n}) \ \forall n$.

Then $K = \{x_1, x_2, \dots\} \subseteq E$. We claim $K' = \{x_0\}$. Clearly, $x_0 \in K'$.

Let $y \neq x_0$, $y \in \mathbb{R}$. We shall prove that $y \notin K'$.

Note that $|y - x_n| \geq |x_0 - x_n| + |y - x_0| \ \forall n \in \mathbb{N}$

$$\geq |y - x_0| - \frac{1}{n}$$

$y \neq x_0 \Rightarrow \exists n_0$ s.t. $|y - x_0| > \frac{2}{n_0}$

$$\Rightarrow |y - x_0| > \frac{2}{n} \ \forall n > n_0$$

$$\Rightarrow |y - x_n| \geq \frac{|y - x_0|}{2} \quad \forall n \geq n_0 \qquad \Rightarrow y \notin K'$$

$\Rightarrow E$ is closed $\Rightarrow E$ is compact. ☐

Try proving this for a general (if it is true) metric space

Note that this gives another necessary and sufficient condition for compactness (every infinite subset must have a limit point in it)

Def. Let (X, d) be a metric space and $Y \subseteq X$. Then $U \subseteq Y$ is said to be open in Y if it is open in the metric space $(Y, d|_Y)$

↳ metric restricted to Y.

or open relative to Y

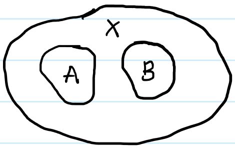
Theo. Let (X, d) be a metric space and $Y \subseteq X$. $U \subseteq Y$ is open in Y iff there is an open set $G \subseteq X$ such that $U = G \cap Y$.

Proof.

Therefore, we can define the subspace topology on $Y \subseteq X$ as $\{Y \cap G : G \subseteq X, G \text{ is open}\}$ which is the set of all open sets in Y considering the metric on X restricted to Y.

Def. Let (X,d) be a metric space. X is said to be disconnected if there exist non-empty $A, B \subseteq X$ such that $A \cup B = X$, $\bar{A} \cap B = \emptyset$, and $A \cap \bar{B} = \emptyset$.
X is said to be connected if it is not disconnected.
Let $E \subseteq X$. E is disconnected if there exist non-empty $A, B \subseteq E$ such that $A \cup B = E$, $\bar{A}^E \cap B = \emptyset$, and $A \cap \bar{B}^E = \emptyset$.

Ex. Let $U \subseteq Y \subseteq X$. Prove that $\bar{U}^Y = Y \cap \bar{U}^X$.



$A \cup B$ is disconnected.

This implies that E is disconnected iff there exist non-empty $A, B \subseteq E$ such that $A \cup B = E$, $\bar{A}^X \cap B = \emptyset$, and $A \cap \bar{B}^X = \emptyset$.

Connectedness seems to correlate to our intuitive understanding of it.

(something like "continuous" sets — intervals?)

Theo. Let $E \subseteq \mathbb{R}$. If E is not an interval, E is disconnected.

Proof. Suppose E is not an interval.

$\rightarrow$ There exist $x, y \in E$ and $z \notin E$ with $x < z < y$. (if there is no such triplet, E is an interval by def.)

Let $A = (-\infty, z) \cap E$ and $B = (z, \infty) \cap E$.

$\bar{A} \subseteq (-\infty, z] \Rightarrow \bar{A} \cap B = \emptyset$

$\bar{B} \subseteq [z, \infty) \Rightarrow A \cap \bar{B} = \emptyset$

$A \cup B = (\mathbb{R} \setminus \{z\}) \cap E = E$

$\Rightarrow E$ is disconnected.

The contrapositive of the above just says:

If $E \subseteq \mathbb{R}$ is connected, E is an interval.

Theo. If $E \subseteq \mathbb{R}$ is an interval, E is connected.

Proof. Suppose E is disconnected. Then $\exists A, B$ s.t. $A \cup B = E$, $\bar{A} \cap B = \emptyset$, and $A \cap \bar{B} = \emptyset$.

Let $x \in A$ and $y \in B$. Assume wlog that $x < y$.

Let $z = \sup(A \cap [x, y])$

$$\Rightarrow z \in \overline{(A \cap [x,y])} \subseteq \bar{A} \cap \overline{[x,y]} = \bar{A} \cap [x,y]$$

$\hookrightarrow$ as $(\overline{A \cap [x,y]})$ is the smallest closed set containing $A \cap [x,y]$

$$\Rightarrow z \in \bar{A} \Rightarrow z \notin B \Rightarrow x \leq z < y.$$

$\hookrightarrow$ as $y \in B$ and $z \in [x,y]$.

(a) If $z \notin A$, then $x < z < y$ and $z \notin E$ (as $z \notin A$ and $z \notin B$). $\Rightarrow E$ is not an interval.

(b) If $z \in A$, then $z \notin \bar{B} = B \cup B'$ (as $A \cap \bar{B} = \emptyset$)

$\Rightarrow \exists r > 0$ s.t. $(z - r, z + r) \cap B = \emptyset$.

$\Rightarrow \forall \tilde{z}$ with $z < \tilde{z} < z + r$, $\tilde{z} \notin B$.

$\Rightarrow \exists z_1 \notin B$, $z_1 > z$ s.t. $z < z_1 < y$

Also, $z_1 \notin A$. ($z = \sup(A \cap [x,y])$ and $x \leq z < y$ and $z_1 > z$)

$\Rightarrow z_1 \notin E$.

Therefore, E is not an interval.

$\therefore$ $E \subseteq \mathbb{R}$ is connected if and only if E is an interval.

Note. This cannot be extended to $\mathbb{R}^n$.
(A circle is connected in $\mathbb{R}^2$.)

Theo. Let (X,d) be a metric space.

1. X is disconnected.
2. $X = C \cup D$ where C, D are non-empty, disjoint, and closed.
3. $X = C \cup D$ where C, D are non-empty, disjoint, and open.
4. There is a non-empty proper subset of X that is both open and closed.

Equivalent.

Proof. $1 \Rightarrow 2$:

Suppose $X = A \cup B$ with $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

We claim A is closed.

$\overline{A} \subseteq X = A \cup B$ and $\overline{A} \cap B = \emptyset \Rightarrow \overline{A} \subseteq A$

$\Rightarrow X = A \cup B$ where A, B are non-empty, disjoint, and closed.

$2 \Rightarrow 1$:

This follows directly.

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04 September 2020 11:01

Def. Let (X,d) be a metric space. X is path-connected if for any $x,y \in X$, there is a continuous function $f:[0,1] \to X$ such that $f(0)=x$ and $f(1)=y$.

Theo. Let (X,d) be a metric space.

X is path-connected $\Rightarrow$ X is connected.

(Converse is not true. Consider $\{(x, \sin\frac{1}{x}) \in \mathbb{R}^2 : x \in \mathbb{R} \setminus \{0\}\}$)

U open connected on $\mathbb{R}^n \Rightarrow$ path-connected