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Sequences

Def. Let (X,d) be a metric space. A sequence in X is a map $f: \mathbb{N} \to X$.

We typically denote sequence as $\{a_1, a_2, \dots\}$, $\{a_n\}_{n\in\mathbb{N}}$ or $(a_n)_{n\in\mathbb{N}}$

Def. A sequence $(a_n)_{n\in\mathbb{N}}$ in (X,d) is convergent if $\exists a \in X$ such that for all $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow d(a_n, a_n) < \varepsilon$$

Equivalently, $a_n \in B(a, \varepsilon)$ $\forall n \geq n_0$

In this case, we say that $(a_n)_{n\in\mathbb{N}}$ converges to a and write $a_n \to a$.

If $(a_n)_{n\in\mathbb{N}}$, we denote the point it converges to by $\lim_{n\to\infty} a_n$

For example, $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ converges to 0 in $\mathbb{R}$.

Theo. If a sequence $(a_n)_{n\in\mathbb{N}}$ converges, its limit is unique.

Proof.

Theo. A convergent sequence is bounded.

(If $(p_n)_{n\in\mathbb{N}}$ is convergent, then $\{p_n : n\in\mathbb{N}\}$ is bounded.)

Proof.

Theo. Let $E\subseteq X$ and a be a limit point of E. Then there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in E such that $a_n \to a$ in X.

Proof. We have $a\in E'$.

For each $n\in\mathbb{N}$, let $a_n \in B(a,\frac{1}{n})\cap E\setminus\{a\} \neq \emptyset$.

We claim that $a_n\to a$ in X.

Indeed, given any $\varepsilon>0$, $\exists n_0\in\mathbb{N}$ such that $\frac{1}{n_0}<\varepsilon$. Then $\forall n\geq n_0$, $d(a_n,a)<\varepsilon$.

Theo. Let $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be convergent in $\mathbb{R}$ such that $s_n\to s$ and $t_n\to t$. If $\forall n\in\mathbb{N}$, $s_n\leq t_n$, then $s\leq t$.

Proof. Suppose $s>t$. Choose some ε such that $0<\varepsilon<\frac{s-t}{2}$. Then let $n_1,n_2\in\mathbb{N}$ such that $\forall n\geq n_1$, $|s_n-s|<\varepsilon$ and $\forall n\geq n_2$, $|t_n-t|<\varepsilon$.

Choose $n_0=\max(n_1,n_2)$.

Then $t_{n_0} < t+\frac{s-t}{2} = \frac{s+t}{2}$ and $s_{n_0} > s-\frac{s-t}{2} = \frac{s+t}{2} \Rightarrow s_{n_0} > t_{n_0}$.

This is a contradiction. Therefore, $s\leq t$.

Corollary. Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}}$ be convergent in $\mathbb{R}$ such that for all $n\in\mathbb{N}$, $a_n\leq b_n\leq c_n$. If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n$, then they are also equal to $\lim\limits_{n\to\infty} b_n$.

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Theo. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be convergent in $\mathbb{R}$.
Then 1. $a_n + b_n \to a+b$

2. $a_n b_n \to ab$

3. $\left(\frac{a_n}{b_n}\right) \to \left(\frac{a}{b}\right)$ if $a_n \neq 0\ \forall n$ and $a \neq 0$.

Proof Sketch. 1. is left as an exercise.

2. $|s_n t_n - st| \leq |s_n||t_n - t| + |t||s_n - s|$
Use the boundedness of $(s_n)_{n\in\mathbb{N}}$.

3. can be proved similar to 2.

Theo. For each $n\in\mathbb{N}$, let $x_n = (\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nk}) \in \mathbb{R}^k$ under the d_2 metric.
Then $x_n \to x = (\alpha_1, \alpha_2, \dots, \alpha_k)$ in $\mathbb{R}^k$ if and only if $\alpha_{ni} \to \alpha_i$ for each i.

Proof. Let $x_n \to x$. $\Rightarrow \forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $n \geq n_0$

$$\Rightarrow d_2(x_n, x) < \varepsilon$$

$$\Rightarrow \sum_{i=1}^{n} (\alpha_{ni} - x_i)^2 < \varepsilon^2 \Rightarrow |\alpha_{ni} - x_i| < \varepsilon$$

$\forall$ valid i and $n \geq n_0$

The converse is similar.

Theo. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in metric space (X,d). Consider a sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ s.t. $n_i < n_j$ for $i<j$.
Then the sequence $(a_{n_k})_{k\in\mathbb{N}}$ is said to be a subsequence of $(a_n)_{n\in\mathbb{N}}$.

The limit of a subsequence is called a subsequential limit.

Ex. Prove that if a sequence is convergent, any subsequence is convergent to the same limit.

A natural extension to the above is: what sequences have convergent subsequences?
For example $(-1,1,-1,1,\dots)$ is not convergent but has a convergent subsequence
$(1,2,3,\dots)$ has no convergent subsequence.

Theo. 1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in a compact metric space (X,d). Then it has a convergent subsequence.
2. Any bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Proof. 1. Consider $E = \{a_1, a_2, \dots\} \subseteq X$ as a set.
If E is finite, then $\exists\, n_1 < n_2 < \dots$ such that $a_{n_1} = a_{n_2} = \dots$.
Then the sequence $(a_{n_k})_{k\in\mathbb{N}}$ is convergent.
Otherwise, let E be an infinite subset of the compact space (X,d)
$\Rightarrow$ E has a limit point a in X (by the Bolzano-Weierstrass Th.)
Then for each $k\in\mathbb{N}$,
let $a_{n_k} \in E \cap B(a,\frac{1}{k}) \setminus \{a\}$
where $n_1 < n_2 < \dots$. This is possible since each neighbourhood of a contains infinitely elements of E.
Then $(a_{n_k})_{k\in\mathbb{N}}$ is convergent.

2. Any bounded sequence in $\mathbb{R}^k$ is contained in a closed and bounded interval, which is compact.
By part 1., the sequence must then have a convergent subsequence.
$\square$

Note that the converse is not true; consider $(1,2,1,3,1,4,1,5,\dots)$

Def. A sequence $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if for every $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$d(p_n, p_m) < \varepsilon \qquad \forall n,m \geq n_0 .$$

Theo. Any convergent sequence is Cauchy.

Proof. Let L be the limit of a convergent seq. $(p_n)_{n\in\mathbb{N}}$. For any $\varepsilon>0$, let $n_0\in\mathbb{N}$ s.t. $d(p_n,L) < \varepsilon/2$ for all $n\geq n_0$.

Then $\forall n,m \geq n_0$, $d(p_n,p_m) \leq d(p_n,L) + d(p_m,L) < \varepsilon$.

The converse is not necessarily true. For example, consider $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}$ given by

$$p_n = 1 + \frac{1}{1!} + \cdots + \frac{1}{n!}$$

Then p_n is Cauchy but not convergent in $\mathbb{Q}$.

Def. A metric space is said to be complete if every Cauchy sequence in it converges.

Lemma. Let $(p_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. If $(p_n)_{n\in\mathbb{N}}$ has a convergent subsequence, then it is convergent.

Suppose $(p_{n_k})_{k\in\mathbb{N}}$ is a convergent subsequence that converges to p.

Then $\exists n_1 : d(p_{n_k}, p) < \varepsilon/2 \quad \forall k \geq n_1$.

$\exists n_2 : d(p_n, p) < \varepsilon/2 \quad \forall k \geq n_2$

$$\Rightarrow \forall n \geq \max(n_1,n_2) \,,\, d(p_n,p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p) < \varepsilon$$

for any $n_k \geq \max(n_1,n_2)$

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Theo. Any compact metric space is complete.

Proof. Let $(p_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in compact metric space (X,d). Then $(p_n)_{n\in\mathbb{N}}$ has a convergent subsequence and the previous lemma implies the result.

Ex. Prove that any Cauchy sequence is bounded.

We can similarly show that $\mathbb{R}^k$ is complete.
(use the result of the above exercise)

Some more limits.

Theo.

1. If $p>0$, $\lim_{n\to\infty} \left(\frac{1}{n}\right)^p = 0$
2. If $p>0$, $\lim_{n\to\infty} p^{1/n} = 1$.
3. $\lim_{n\to\infty} n^{1/n} = 1$.
4. If $p>0$, for $\alpha\in\mathbb{R}$, $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.
5. If $|x|<1$, then $\lim_{n\to\infty} x^n = 0$.

Proof.

1. For each $\varepsilon>0$, just choose $n_0 = \left\lceil (1/\varepsilon)^{1/p} \right\rceil$.

2. If $p>1$, $x_n = p^{1/n} - 1 > 0$ for each n.

   Note that $p = (1+x_n)^n \geq nx_n$

   $\Rightarrow 0 < x_n \leq \frac{p}{n}$

   $\Rightarrow \lim_{n\to\infty} x_n = 0 \Rightarrow \lim_{n\to\infty} p^{1/n} = 1$.

   If $0<p<1$, start with the fact that $\lim_{n\to\infty} (1/p)^{1/n} = 1$

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3. Let $x_n = n^{1/n} - 1 \geq 0$.

Then $n = (1+x_n)^n \geq \frac{n(n-1)}{2} x_n^2$ (Same idea as 2)

$$\Rightarrow 0 \leq x_n \leq \sqrt{\frac{2}{n-1}} \Rightarrow \lim_{n\to\infty} x_n = 1.$$

4. Let $k \in \mathbb{N}$ such that $k > \alpha$.

For $n > 2k$,

$$(1+p)^n \geq {}^nC_k\, p^k = \frac{p^k}{k!}\left(n \cdot (n-1) \cdot \cdots \cdot (n-k+1)\right)$$

$$\geq \frac{p^k}{k!} \cdot \left(\frac{n}{2}\right)^k$$

$$\Rightarrow \frac{n^\alpha}{(1+p)^n} \leq n^{\alpha-k} \cdot 2^k \cdot \frac{2}{p^k}$$

Then use (1) to get $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

5. Set $\alpha = 0$ and $p = \left(\frac{1}{|x|} - 1\right)$ in (4).

lim sup and lim inf

We define the extended real number line by $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ where

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Def. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence.

1. We write $a_n \to \infty$ if for all $\alpha > 0$, there exists $n_0 \in \mathbb{N}$ such that
$$a_n > \alpha \text{ for all } n > n_0.$$
2. We write $a_n \to -\infty$ if $-a_n \to \infty$.

Def. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of reals and let

$$E = \{x \in \overline{\mathbb{R}} : x = \lim_{k\to\infty} s_{n_k} \text{ for some subsequence } (s_{n_k})_{k\in\mathbb{N}} \text{ of } (s_n)_{n\in\mathbb{N}}\}$$

(If $s_{n_k} \to \infty$, we take its lim as ∞
If $s_{n_k} \to -\infty$, we take its lim as $-\infty$)

We then define $\limsup_{n\to\infty} s_n = \sup E$ and $\liminf_{n\to\infty} s_n = \inf E$.

For example, if $s_n = (-1)^n\left(1+\frac{1}{n}\right)$, $E = \{1,-1\}$ so $\limsup_{n\to\infty} s_n = 1$ and $\liminf_{n\to\infty} s_n = -1$.

Let $a_n = \frac{1}{n}$ if n is odd and n if n is even. Then $E = \{0, \infty\}$ so the lim sup is ∞ and the lim inf is 0.

What if $(s_n)_{n\in\mathbb{N}}$ is convergent? Then since any subsequence converges to the limit, the lim, lim sup, and lim inf are equal.
What about the converse?

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Lemma. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of reals such that $t > \limsup_{n\to\infty} s_n$. There then exists $n_0 \in \mathbb{N}$ such that $s_n < t$ for all $n \geq n_0$.

Proof. If there exists no n_0, then there are infinitely many n such that $s_n \geq t$. If this set is unbounded, then $\to \infty$. Otherwise, there is a subsequence with limit $\geq t$. This is a contradiction to the fact that the lim sup is the supremum of the set of subsequential limits. $\square$

Def. A sequence $(a_n)_{n\in\mathbb{N}}$ of reals is said to be

monotonically increasing if $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$.

monotonically decreasing if $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$.

Theo. A monotonically increasing bounded sequence converges to its supremum (as a set).

(Left as exercise)

Theo. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of reals. Then

$$\limsup_{n\to\infty} s_n = \inf_{n\to\infty} \sup\{s_m : m \geq n\}$$

$$\liminf_{n\to\infty} s_n = \sup_{n\to\infty} \inf\{s_m : m \geq n\}$$

(This follows directly from the lemma two above)

Note that the above implies

$$t < \limsup_{n\to\infty} s_n \Rightarrow t < \sup\{s_m : m \geq n\} \ \forall n$$

$$\Rightarrow t < s_n \text{ for infinitely many } n.$$

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Series

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of reals.

Def. We say $\sum_{n=1}^{\infty} a_n$ converges if $\left(\sum_{k=1}^{n} a_k\right)_{n\in\mathbb{N}}$ converges. We then say that it is equal to the limit of this sequence.

For example, if $|x|<1$, then defining $S_n = 1+x+\dots+x^n = \frac{1-x^n}{1-x}$, we see that $\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$.

Theo: $\sum_{n=1}^{\infty} a_n$ converges if and only if for any $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that $\left|\sum_{k=n}^{m} a_n\right| < \varepsilon$ for all $n,m \geq n_0$.

Proof. Letting $S_n = \sum_{k=1}^{n} a_n$, we see that the sum converges iff $(S_n)_{n\in\mathbb{N}}$ is Cauchy, which is equivalent to the given condition.

If Σa_n does not converge, we say that it diverges.

Theo.
1. If $|a_n| \leq c_n$ $\forall n \geq n_0$ and Σc_n converges, then Σa_n converges.
2. If $a_n \geq d_n \geq 0$ and Σd_n diverges, then Σa_n diverges.

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Theo. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p \geq 0$ iff $p > 1$.

Proof. For $p > 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \leq 1 + \frac{1}{2^p} + \frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{8^p} + \cdots$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \cdots \quad \text{which converges as } p > 1.$$

For $p = 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$\geq \frac{1}{2} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots$$

$$\geq 1 + 1 + 1 + \cdots \quad \text{which diverges.}$$

For $p < 1$,

This follows directly as $\frac{1}{n^p} \geq \frac{1}{n}$ for $p < 1$. $\square$

If $a_n \geq 0$ for all n and $\left(\sum_{k=1}^{n} a_n\right)_{n \in \mathbb{N}}$ is bounded, then $\sum_{n=1}^{\infty} a_n$ converges.

(as the seq. is monotonic increasing and bounded above)

Theo. [Cauchy's Criterion]

If $a_n \geq 0$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

Try showing that either both are bounded or both are unbounded.

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Ex. Show that $\sum_{n=2}^{\infty} \frac{1}{n \cdot (\log_2 n)^p}$ converges iff $p > 1$.

Theo. [Root Test]

For $a_n \in \mathbb{R}$, let $\alpha = \limsup |a_n|^{1/n}$.

If $\alpha < 1$, then $\sum_{n=1}^{\infty} a_n$ converges and if $\alpha > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Suppose $\alpha < 1$. Let $\beta : \alpha < \beta < 1$.

$\limsup |a_n|^{1/n} < \beta$

$\Rightarrow \exists n_0$ such that $\forall n \geq n_0$, $|a_n|^{1/n} < \beta$

$\Rightarrow |a_n| < \beta^n$.

However, $\sum_{n=1}^{\infty} \beta^n$ converges which implies that $\sum_{n=1}^{\infty} a_n$ converges.

If $\alpha > 1$, then there are infinitely many n such that $|a_n|^{1/n} > 1$

$\Rightarrow |a_n| > 1$.

This implies $\sum_{n=1}^{\infty} a_n$ diverges. $\square$

Note that the root test is inconclusive if $\alpha = 1$.

Consider $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$.

Theo. [Ratio Test]

Let $\alpha = \limsup \left| \frac{a_{n+1}}{a_n} \right|$.

If $\alpha < 1$, then $\sum a_n$ converges.

If there exists n_0 such that $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$ for all $n \geq n_0$, then $\sum a_n$ diverges.

Proof. Again choose β such that $\alpha < \beta < 1$.

There then exists n_0 such that $\forall n \geq n_0$, $|a_{n+1}| < \beta |a_n|$

$< \beta^{n-n_0+1} |a_{n_0}|$.

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We define e as

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

(this converges as $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$)

We then have that $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$.

Proof

$$\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \leq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

$$\Rightarrow \limsup \left(1+\frac{1}{n}\right)^n \leq e.$$

Fixing some m, for any $n \geq m$

$$\left(1+\frac{1}{n}\right)^n \geq 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)$$

$$\liminf \left(1+\frac{1}{n}\right)^n \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!} \quad \forall m.$$

$$\Rightarrow \liminf \left(1+\frac{1}{n}\right)^n \geq e$$

However, as $\underset{\geq e}{\underset{\downarrow}{\liminf}} \leq \underset{\leq e}{\underset{\downarrow}{\limsup}}$, this implies that the limit is e.