Sequences and Series of Functions

Def. Let (X,d) be a metric space and $E \subseteq X$. For each $n \in \mathbb{N}$, let $f_n : E \to \mathbb{R}$ (or $\mathbb{C}$). Suppose that for each $x \in E$, $(f_n(x))_{n\in\mathbb{N}}$ converges. Then $f(x) = \lim_{n\to\infty} f_n(x)$ is the limit function.

For example, if $f_n : [0,1] \to \mathbb{R}$ given by $x \mapsto x^n$, then the limit

$$f(x) = \begin{cases} 0, & 0<x<1 \\ 1, & x=1 \end{cases}$$

Similarly, if $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions in E and $\sum_{n=1}^{\infty} f_n(x)$ converges for each x, then $f(x) = \sum_{i=1}^{\infty} f_n(x)$ is well-defined.

Now suppose we know that $(f_n)_{n\in\mathbb{N}} \to f$ and each f_n is continuous. Then is f continuous?

No! Consider the example given above where $f_n : [0,1] \to \mathbb{R}$ with $f_n(x) = x^n$.

If each f_n is differentiable, f need not be differentiable either.

Same example as before.

If f is differentiable, then does $f_n' \to f'$?

Again, no! Consider $f_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}$. Then $f(x) = 0$.

However, $f_n' = \{\sqrt{n}\sin(nx)\}$ does not converge.

What about Riemann integrability?

No! Let $f_n(x) = n^2 x(1-x^2)^n \ : 0 \le x \le 1$.

$f(x) = 0$ everywhere.

For each n, $\displaystyle\int_0^1 f_n(x)\,dx = \frac{n^2}{2(n+1)}$ diverges!

Let $\mathbb{Q} \cap [0,1] = \{r_1, r_2, \dots\}$, $f_n(x) = \begin{cases} 1 & x \in \{r_1,\dots,r_n\} \\ 0 & \text{otherwise.} \end{cases}$ and $f_n \to f$. Then each f_n is Riemann integrable but f is not.

We must modify our definition of convergence for all this to hold.

Def. Let (X,d) be a metric space and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from $E \subseteq X$ to $\mathbb{R}$ (or $\mathbb{C}$). This sequence is said to converge uniformly to $f(x)$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in E.$$

Let $f_n : [0,1] \to \mathbb{R}$ with $f_n(x) = x^n$ and $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$

$$|f_n(x) - f(x)| = \begin{cases} |x^n|, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

For any $\varepsilon > 0$, we must find $N \in \mathbb{N}$ s.t. $|x^n| < \varepsilon$ for all $n \ge N$ and $0 \le x < 1$. Such an N does not exist.
f_n does not converge uniformly to f!

$f_n : [0,1] \to \mathbb{R}$ given by $f_n(x) = \frac{x}{n}$ converges to $f : [0,1] \to \mathbb{R}$ with $f(x) = 0$ $\forall x$.

We similarly define uniform convergence of the partial sum of functions.

Theo. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions on E that converges point-wise to f. Let $M_n = \sup\limits_{x \in E} |f_n(x) - f(x)|$.
f_n converges uniformly to f if and only if $M_n \to 0$.

Proof. Suppose $M_n \to 0$. Then there exists N such that for all $n \ge N$,

$$M_n < \varepsilon \Rightarrow \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon$$

$$\Rightarrow |f_n(x) - f(x)| < \varepsilon \text{ for all } x \in E$$

$$\Rightarrow f_n \to f \text{ uniformly.}$$

Conversely, let $N \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < \varepsilon/2 \quad \text{for all } x \in E \text{ and } n \ge N$$

$$\Rightarrow \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \ge N$$

$$\Rightarrow M_n \to 0$$

Theo. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in E and let $\sum_{n=1}^{\infty} f_n \to f$ (pointwise). Suppose $|f_n(x)| < M_n$ for all x and $\sum_{n=1}^{\infty} M_n < \infty$.

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly to f.

Proof. For $x \in E$, let $s_n(x) = \sum_{i=1}^{n} f_i(x)$

Then
$$|s_n(x) - f(x)| = \left|\sum_{m=n+1}^{\infty} f_m(x)\right|$$
$$\leq \sum_{m=n+1}^{\infty} |f_m(x)|$$
$$\leq \sum_{m=n+1}^{\infty} M_n = A_n$$

Then $\sup_{x\in E} |s_n(x) - f(x)| \leq A_n$

However, $A_n \to 0 \Rightarrow \sup_{x\in E} |s_n(x) - f(x)| \longrightarrow 0$.

$$\Rightarrow \sum_{n=1}^{\infty} f \to f \text{ uniformly.}$$

Theo. [Cauchy's Theorem for Uniform Convergence]

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of complex-valued functions on $E \subseteq \mathbb{R}$. Then $(f_n)_{n\in\mathbb{N}}$ converges uniformly if and only if for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that
$$n, m \geq N \Rightarrow |f_n(x) - f_m(x)| < \varepsilon \text{ for all } x \in E.$$

Proof. Suppose $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

For $\varepsilon > 0$, $\exists N$ s.t. for all $m, n > N$
$$|f_n(x) - f(x)| < \varepsilon/2 \quad \text{and} \quad |f_m(x) - f(x)| < \varepsilon/2$$
$$\Rightarrow |f_m(x) - f_n(x)| < \varepsilon$$

Conversely, if for $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $m, n > N$
$$\Rightarrow |f_m(x) - f_n(x)| < \varepsilon \text{ for all } x \in E.$$

As for a fixed $x \in E$, $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy, it converges to some $f(x)$.

$\Rightarrow |f_n(x) - f(x)| \leq \varepsilon$ for all $n > N$ and $x \in E$ and $f_n \to f$ uniformly.

Theo. Let f_n be a sequence of functions in E and let x be a limit point of E. Suppose f_n converges uniformly to f on E. Then

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t) \quad \text{(provided both sides exist)}$$

(needn't always be the case,

$$\lim_{x\to 0}\lim_{n\to\infty}\frac{nx}{1+nx} = 1 \neq 0 = \lim_{n\to\infty}\lim_{x\to 0}\frac{nx}{1+nx}\text{)}$$

Corollary. If f_n is continuous for each $n\in\mathbb{N}$ and $f_n \to f$ uniformly, then f is continuous.

Proof. Let $\lim_{t\to x} f_n(t) = A_n$.

We shall prove:
- A_n converges
- If $A_n \to A$, then $\lim_{t\to x} f(t) = A$

For $\varepsilon>0$, $\exists N\in\mathbb{N}$ s.t. for all $m,n>N$ and $t\in E$, $|f_n(t)-f_m(t)|<\varepsilon$.
Then

$$\left|\lim_{t\to x} f_n(t) - \lim_{t\to x} f_m(t)\right| \le \varepsilon$$

$$|A_n - A_m| \le \varepsilon \quad \Rightarrow (A_n)_{n\in\mathbb{N}} \text{ converges.}$$

Let $A_n \to A$.

Now,

$$|f(t)-A| \le |f(t)-f_n(t)| + |f_n(t)-A_n| + |A_n - A|$$
$$< \varepsilon+\varepsilon+\varepsilon = 3\varepsilon$$

if $0<d(t,x)<\delta$

where n satisfies $|A_n - A|<\varepsilon$ $\quad (A_n\to A)$

$|f(t)-f_n(t)|<\varepsilon$ $\quad (f_n\to f)$

and $d(t,x)<\delta \Rightarrow |f_n(t)-A_n|<\varepsilon$

$\left(\lim_{t\to x} f_n(t) = A_n\right)$

Change ε to $\varepsilon/3$.

$\Rightarrow \lim_{t\to x} f(t) = A$.

Theo. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Riemann integrable functions on $[a,b]$ that converges uniformly to f. Then f is Riemann integrable and

$$\int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx$$

Proof. Let $\varepsilon_n = \sup\limits_{x\in[a,b]} |f_n(x) - f(x)|$

Since $f_n \to f$ uniformly, $\varepsilon_n \to 0$.

$$f_n(x) - \varepsilon_n \le f(x) \le f(x) + \varepsilon_n$$

$\forall x \in [a,b]$ and $n\in\mathbb{N}$.

Now,

$$\int_a^b (f_n(x) - \varepsilon_n)\,dx \le \int_a^b (f_n(x) + \varepsilon_n)\,dx$$

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le 2\varepsilon_n \int_a^b dx = 2\varepsilon_n(b-a) \quad \text{for all } n.$$

As $2\varepsilon_n(b-a) \to 0$, f is integrable.

We also have

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \le \int_a^b |f_n(x) - f(x)|\,dx \le \varepsilon_n(b-a)$$

As $\varepsilon_n \to 0$,

$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$$

Corollary. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Riemann integrable functions on $[a,b]$ such that $\sum\limits_{k=1}^{n} f_k(x)$ converges uniformly on $[a,b]$. Then $\sum\limits_{n=1}^{\infty} f_n$ is Riemann integrable and

$$\sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx = \int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx$$

(Let $S_n = \sum\limits_{k=1}^{n} f_n$. Each S_n is Riemann integrable and S_n converges uniformly)

Theo. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable functions on $[a,b]$. Let $(f_n(x_0))_{n\in\mathbb{N}}$ be convergent for some $x_0 \in [a,b]$. Further assume that $(f_n')_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$. Then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to some function f and $f_n' \to f'$.

Proof. For $\varepsilon>0$, choose $N\in\mathbb{N}$ s.t. $\forall n,m\in\mathbb{N}$,

$$|f_n(x_0) - f_m(x_0)| < \varepsilon/2$$
$$\text{and } |f_n'(t) - f_m'(t)| < \varepsilon/2(b-a) \quad \text{for all } t\in[a,b].$$

Then for any $x\in[a,b]$,

$$\left|(f_n(x)-f_m(x)) - (f_n(t)-f_m(t))\right| < |x-t| \cdot \left(\varepsilon/2(b-a)\right) \leq \varepsilon/2$$

(Applying MVT on $f_n - f_m$)

$$\Rightarrow |f_n(x) - f_m(x)| < \varepsilon$$

So by Cauchy's criterion, $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[a,b]$.
Let $f_n \to f$.
Now, we claim that f is differentiable and $f'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x\in[a,b]$

Fix some $x\in[a,b]$.
Define
$$\varphi_n(t) = \frac{f_n(t) - f_n(x)}{t-x}, \quad t\in[a,b]\setminus\{x\}$$

$$\varphi(t) = \frac{f(t)-f(x)}{t-x}$$

As each f_n is differentiable, $\lim_{t\to x} \varphi_n(t) = f_n'(x)$

Now,
$$|\varphi_n(t) - \varphi_m(t)| = \frac{1}{|t-x|}\left|(f_n(x)-f_m(x)) - (f_n(t)-f_m(t))\right|$$
$$< \frac{1}{|t-x|}\cdot|t-x|\cdot \varepsilon/2(b-a) \quad \text{for all } m,n>N$$

$\to \varphi_n$ converges uniformly (to φ, in fact)

Finally, $\lim_{t\to x}\lim_{n\to\infty}\varphi_n(t) = \lim_{n\to\infty}\lim_{t\to x}\varphi_n(t)$

$$f'(x) = \lim_{t\to x}\varphi(t) = \lim_{n\to\infty} f_n'(x)$$

This completes the proof.

Power Series

Def A series of the form $\sum_{k=0}^{n} a_k (x-\alpha)^k$ where $\alpha \in \mathbb{R}$ and $a_n \in \mathbb{C}$ for each n is known as a power series around α.

When does a power series converge?

WLOG, let $\alpha = 0$.

By the root test, it converges for $\limsup_{n\to\infty} |a_n x^n|^{1/n} < 1$

$$\Rightarrow |x| < \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} = R$$

The series diverges if $|x| > R$.

We do not know what happens when $|x| = R$.

Def. Given a power series $\sum_{k=1}^{n} a_n (x-\alpha)^n$, its radius of convergence is defined as

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

If $\limsup_{n\to\infty} |a_n|^{1/n} = 0$, then $R = \infty$. (Consider $\sum \frac{x^n}{n!}$)

If $\limsup_{n\to\infty} |a_n|^{1/n} = \infty$, then $R = 0$ (Consider $\sum n^n x^n$)

Theo. Let R be the radius of convergence of $f(x)=\sum a_n x^n$.

1. The series $\sum_{k=0}^{n} a_k x^k$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$ for any $\varepsilon>0$.
2. $\sum_{n=1}^{\infty} n a_n x^{n-1}$ also has radius of convergence R. Further,

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Proof

1. For $|x| \le (R-\varepsilon)^n$, $\quad \sum_{n=0}^{\infty} a_n |x|^n \le \underbrace{\sum_{n=0}^{\infty} a_n (R-\varepsilon)^n}_{\text{converges}}$

$$\rightarrow \text{It converges uniformly.}$$

2. Consider $\quad \limsup_{n\to\infty} |n a_n|^{1/n} = \lim_{n\to\infty} n^{1/n} \limsup_{n\to\infty} |a_n|^{1/n}$

$$= 1 \times \limsup_{n\to\infty} |a_n|^{1/n}$$

$\Rightarrow$ Same radius of convergence.

The fact that this is the derivative arises by considering $\left(\sum_{k=0}^{n} a_k x^k\right)_{k\in\mathbb{N}}$ and $\left(\sum_{k=1}^{n} k a_k x^{k-1}\right)_{k\in\mathbb{N}}$ which converge uniformly on $|x|\le R-\varepsilon$. Use the Theorem on Page 6 to get the result.

Corollary Suppose $\sum a_n x^n = \sum b_n x^n$ for $|x|<R$ for some $R>0$. Then $a_n=b_n$ for all n.

Now, suppose we have $(a_{ij})_{i\in\mathbb{N}, j\in\mathbb{N}}$. When are the two sums $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ equal?

To see that they need not always be equal consider

$$\begin{array}{cccc} -1 & 0 & 0 & 0 & \cdots \\ 1/2 & -1 & 0 & 0 \\ 1/2^2 & 1/2 & -1 & 0 \\ 1/2^3 & 1/2^2 & 1/2 & -1 \\ & & \vdots & \end{array}$$

That is,

$$a_{ij} = \begin{cases} -1, & i=j \\ 0, & j>i \\ \dfrac{1}{2^{i-j}}, & i>j \end{cases}$$

Then $\sum_i \sum_j a_{ij} = -2$

$\sum_j \sum_i a_{ij} = 0$

Theo. Let (a_{ij}) be a sequence of reals such that $\sum_{i=1}^{\infty} b_i < \infty$ for $b_i = \sum_{j=1}^{\infty} |a_{ij}|$. Then

Dominated Convergence Theorem

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}.$$

Proof- Let $E = \{x_0, x_1, \ldots\}$ be a countable set such that $x_n \to x_0$.
Define $f_i : E \to \mathbb{R}$ by

$$f_i(x_0) = \sum_{j=1}^{\infty} a_{ij} \quad \text{and} \quad f_i(x_n) = \sum_{j=1}^{n} a_{ij}$$

and $g : E \to \mathbb{R}$ by

$$g(x) = \sum_{i=1}^{\infty} f_i(x) \quad \text{(provided it exists)}$$

Note that f_i is continuous on E ($E\setminus\{x_0\}$ consists only of isolated points and it is continuous at x_0)

Also, $\sum f_i$ converges uniformly to g.

$$\left(|f_i(x)| \le \sum_{j=1}^{\infty} |a_{ij}| = b_i \text{ and } \sum b_i < \infty \right)$$

$\Rightarrow g$ is continuous.

Now,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} f_i(x_0) = g(x_0)$$

$$= \lim_{n\to\infty} g(x_n) \qquad (g \text{ is continuous})$$

$$= \lim_{n\to\infty} \sum_{i=1}^{\infty} f_i(x_n)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{n} a_{ij}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{ij}$$ (we can exchange because one of them is a finite sum)

$$= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

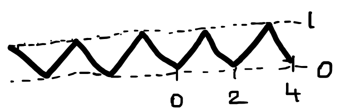
Corollary. If $a_{ij} \ge 0$, then $\sum_j \sum_i a_{ij} = \sum_i \sum_j a_{ij}$ (if both sides are finite)

Theorem. There exists a function f on $\mathbb{R}$ that is continuous everywhere but nowhere differentiable.

Proof. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(x) = |x| \quad \text{on } [-1, 1]$$

$$\varphi(x+2) = \varphi(x) \quad \text{for all } x \in \mathbb{R}$$



For all $s, t \in \mathbb{R}$, $|\varphi(s) - \varphi(t)| \le |s-t|$

$\Rightarrow \varphi$ is (uniformly) continuous on $\mathbb{R}$.

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x) \quad \text{on } \mathbb{R}$$

$\varphi(4^n x)$ is not differentiable for $x \in \frac{1}{4^n}\mathbb{Z}$.

The series we consider in f is convergent because $0 \le \varphi(x) \le 1$.
In fact, it is uniformly convergent because $\left(\frac{3}{4}\right)^n \varphi(4^n x) \le \left(\frac{3}{4}\right)^n$
and $\sum \left(\frac{3}{4}\right)^n < \infty$.
As $\left(\frac{3}{4}\right)^n \varphi(4^n x)$ is continuous, f is continuous as well.

Now, fix some $x \in \mathbb{R}$ and $m \in \mathbb{N}$.
Let $\delta_m = \pm \frac{1}{2} 4^{-m}$, where the sign is appropriately chosen to ensure that there is no integer between $4^m x$ and $4^m(x+\delta_m)$.
(possible because $4^m \delta_m = \frac{1}{2}$)

Let $\gamma_n = \dfrac{\varphi_m(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}$

$$\left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| = \left| \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \cdot \gamma_n \right|$$

If $n > m$, $\gamma_n = \dfrac{\varphi(4^n x + 4^n \delta_m) - \varphi(4^n x)}{\delta_m}$

$= 0$. (In this case, $4^n \delta_m = \pm \frac{1}{2} 4^{n-m}$ is even)

$$\Rightarrow \left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| = \left| \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n \right|$$

If $0 \le n \le m$,

$$\begin{aligned} |\gamma_n| &= 2 \cdot 4^m \cdot |\varphi(4^n x + 4^n \delta_m) - \varphi(4^n x)| \\ &\le 2 \cdot 4^m \cdot 4^n \delta_m \qquad (|\varphi(s) - \varphi(t)| \le |s-t|) \\ &= 4^n \end{aligned}$$

Also, in particular,

$$\begin{aligned} \left(\frac{3}{4}\right)^m \gamma_m &= \left(\frac{3}{4}\right)^m \cdot \frac{1}{\delta_m} \left(\varphi(4^m x \pm \tfrac{1}{2}) - \varphi(4^m x) \right) \\ &= \left(\frac{3}{4}\right)^m \cdot \frac{1}{\delta_m} \cdot \left(\pm \frac{1}{2}\right) \end{aligned}$$

(where the sign is the same as that in δ_m)

(this follows because there is no integer in that interval)

$$= 3^m$$

Now,

$$\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| = \left|\sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n \gamma_n + \left(\frac{3}{4}\right)^m \gamma_m\right|$$

$$\geq \left(\frac{3}{4}\right)^m \gamma_m - \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n \gamma_n$$

$$\geq 3^m - \sum_{n=0}^{m-1} 3^n$$

$$= 3^m - \left(\frac{3^m-1}{2}\right) = \frac{3^m+1}{2}$$

This goes to ∞ as $m \to \infty$ (and $\delta_m \to 0$).

$\Rightarrow f$ is not differentiable at any x.

Theo. [Weierstrass Approximation Theorem]

Let f be a continuous function on a closed and bounded interval f. There is then a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ such that $p_n \to f$ uniformly on $[a,b]$.

Proof. WLOG, assume $[a,b] = [0,1]$. (appropriately shift and scale to get $[a,b]$)

We can also assume $f(0)=f(1)=0$.

(For any general continuous function g on $[0,1]$, consider $f(x) = g(x) - g(0) - x(g(1)-g(0))$. We can then approximate f by polynomials and add $g(0) + x(g(1)-g(0))$ to each to get an approximation of f.)

Now, extend f to $\mathbb{R}$ by $f(x)=0$ if $x \notin [0,1]$.

Then f is uniformly continuous.

(It is uniformly continuous on $[0,1]$ as continuity on a closed and bounded interval implies uniformly continuity and it is trivially uniformly continuous elsewhere.)

Define $Q_n(x) = c_n(1-x^2)^n$, where c_n is such that $\int_{-1}^{1} Q_n(t)\,dt = 1$.

Now, $$\int_{-1}^{1}(1-x^2)^n \geq 2\int_0^{1/\sqrt{n}}(1-t^2)^n dt \geq 2\int_0^{1/\sqrt{n}}(1-nt^2)\,dt = \frac{4}{3}\cdot\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}}$$

$\Rightarrow c_n < \sqrt{n}$

Note that for any $\delta > 0$ and $x: \delta \le |x| < 1$,

$$Q_n(x) \le c_n (1-\delta^2)^n$$
$$< \sqrt{n}(1-\delta^2)^n$$

$\Rightarrow Q_n(x) \to 0$ as $n \to \infty$ uniformly on $x: \delta \le |x| < 1$

Now, define

$$p_n(x) = \int_{-1}^{1} f(x+t)\, Q_n(t)\, dt$$

$$p_n(x) = \int_{-x}^{1-x} f(x+t)\, Q_n(t)\, dt \qquad (f \text{ is } 0 \text{ outside of } [0,1])$$

$$p_n(x) = \int_0^1 f(y)\, Q_n(y-x)\, dy$$

This is a polynomial in x.

(this is more obvious if we write Q_n as a polynomial $\sum_i a_i x^i$)

We claim $p_n \to f$ uniformly on $[0,1]$.

Fix $\varepsilon > 0$. There is $\delta > 0$ s.t. $|y-x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon/2$

Let $M = \sup_{x \in \mathbb{R}} f(x)$.

Then, $$|p_n(x) - f(x)| = \left| \int_{-1}^{1} f(x+t)\, dt\, Q_n(t) - f(x) \int_{-1}^{1} Q_n(t)\, dt \right|$$

$$= \left| \int_{-1}^{1} (f(x+t) - f(x))\, Q_n(t)\, dt \right|$$

$$\le \int_{-1}^{1} |f(x+t) - f(x)|\, Q_n(t)\, dt$$

$$= \int_{-1}^{-\delta} |f(x+t) - f(x)|\, Q_n(t)\, dt + \int_{-\delta}^{\delta} |f(x+t) - f(x)|\, Q_n(t)\, dt$$
$$+ \int_{\delta}^{1} |f(x+t) - f(x)|\, Q_n(t)\, dt$$

$$\le 2M\sqrt{n}(1-\delta^2)^n \cdot (1-\delta) + \frac{\varepsilon}{2} \cdot 1 + 2M \cdot \sqrt{n}(1-\delta^2)^n \cdot (1-\delta)$$

$$\le 4M\sqrt{n}(1-\delta^2)^n + \frac{\varepsilon}{2}.$$

$\left(\int_{-\delta}^{\delta} Q_n \le \int_{-1}^{1} Q_n = 1 \right)$

For large enough n, this is $< \varepsilon$.

Theo. $(C(X), d)$ is a complete metric space.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be Cauchy in $C(x)$.
For $\varepsilon>0$, $\exists N\in\mathbb{N}$ s.t.

$$d(f_n, f_m) < \varepsilon \quad \text{for all } m,n > N.$$

$$\Rightarrow \sup_{x\in X} |f_n(x) - f_m(x)| < \varepsilon$$

$\Rightarrow (f_n)_{n\in\mathbb{N}}$ converges uniformly on X, to say f.

As each f_n is continuous, f is continuous and is in $C(f)$.

$\Rightarrow f \in C(X)$ and $d(f_n, f) \to 0$ as $n\to\infty$

$\Rightarrow (C(X), d)$ is complete.

Def. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of (complex-valued) functions on E. $(f_n)_{n\in\mathbb{N}}$ is pointwise bounded if for each $x\in E$, $(f_n(x))_{n\in\mathbb{N}}$ is bounded. It is said to be uniformly bounded if there exists $M>0$ such that $|f_n(x)|\le M$ for all $n\in\mathbb{N}$ and $x\in E$.

Now, similar to the Bolzano-Weierstrass theorem, we ask the following question:
Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of (complex-valued) functions that is pointwise bounded (or even uniformly bounded). Does it contain a convergent subsequence?
No.
Consider $f_n(x)=\sin(nx)$ on $[0,2\pi]$.
Clearly, each f_n is continuous and the sequence is uniformly bounded.
If it has a convergent subsequence, say $(f_{n_k})_{k\in\mathbb{N}}$, then

$$\lim_{k\to\infty}(\sin n_{k+1}x-\sin n_k x)=0$$

$$\int_0^{2\pi}\lim_{k\to\infty}(\sin n_{k+1}x-\sin n_k x)^2=0$$

$$\underbrace{\lim_{k\to\infty}\int_0^{2\pi}(\sin n_{k+1}x-\sin n_k x)^2}_{=2\pi.}=0$$

(Using something called the Dominated Convergence Theorem in measure theory)

Contradiction! It has no convergent subsequence despite being (uniformly) bounded.

(Note that we do not require the DCT to assert that there is no uniformly convergent subsequence.)

Def. Let $\mathcal{F}$ be a family of (complex-valued) functions on E. $\mathcal{F}$ is said to be equicontinuous if for all $\varepsilon>0$, there exists $\delta>0$ s.t.

$$d(x,y)<\delta \text{ and } x,y\in E \implies |f(x)-f(y)|<\varepsilon \quad \text{for all } f\in\mathcal{F}.$$

(δ is independent of both x,y and $f\in\mathcal{F}$)

If $\mathcal{F}$ is equicontinuous, each $f\in\mathcal{F}$ is uniformly continuous.

Theo. Let K be a compact metric space and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C(K)$ such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on K. Then $(f_n)_{n\in\mathbb{N}}$ is equicontinuous on K.

Proof. For $\varepsilon>0$, there is $N\in\mathbb{N}$ s.t.

$$d(f_n,f_N) := \sup_{x\in X} |f_n(x)-f_N(x)| < \varepsilon/3 \qquad \text{— (1)}$$

for all $n>N$

Each f_n is continuous on K so it is also uniformly continuous.
For $1\le i\le N$, there is $\delta>0$ s.t.

$$\underset{x,y\in K}{d(x,y)<\delta} \Rightarrow |f_i(x)-f_i(y)| < \varepsilon/3 . \qquad \text{— (2)}$$

(because there are finitely many i, just take the minimum of the N δ_i's)

For $n>N$ and $d(x,y)<\delta$, (and $x,y\in K$)

$$\begin{aligned} |f_n(x)-f_n(y)| &\le |f_n(x)-f_N(x)| + |f_N(x)-f_N(y)| + |f_N(y)-f_n(y)| \\ &< \underset{(1)}{\varepsilon/3} + \underset{(2)}{\varepsilon/3} + \underset{(1)}{\varepsilon/3} \\ &= \varepsilon \end{aligned} \qquad \text{— (3)}$$

From (2) and (3), $(f_n)_{n\in\mathbb{N}}$ is equicontinuous. ■

Consider $f_n(x)=x^n$ on $[0,1]$. $(f_n)_{n\in\mathbb{N}}$ is NOT equicontinuous.

We would need $|x^n-1|<\varepsilon$ if $1-\delta<x<1$ for all n.
$(f_n)_{n\in\mathbb{N}}$ is not uniformly continuous.

Revisiting the question from the previous page,

Lemma Let E be a countable set and $(f_n)_{n\in\mathbb{N}}$ be a pointwise bounded sequence of functions on E. Then $(f_n)_{n\in\mathbb{N}}$ has a pointwise convergent subsequence.

Proof. Let $E=\{x_1,x_2,\dots\}$. $(f_n(x_1))_{n\in\mathbb{N}}$ is a bounded sequence of complex numbers so it has a convergent subsequence, say $(f_{1,k}(x_1))_{k\in\mathbb{N}}$.

So,

S_1: $f_{1,1}$ $f_{1,2}$ $f_{1,3}$ $\cdots$ converges at x_1.

Consider $(f_{1,k}(x_2))_{k\in\mathbb{N}}$, a bounded sequence.

It has a convergent subsequence $(f_{2,k}(x_2))_{k\in\mathbb{N}}$.

S_2: $f_{2,1}$ $f_{2,2}$ $f_{2,3}$ $\cdots$ converges at x_1 and x_2.

$(S_2 \subseteq S_1)$

Continue this to get for each $n\in\mathbb{N}$, a sequence

S_n: $f_{n,1}$ $f_{n,2}$ $f_{n,3}$ $\cdots$ which converges at $x_1, x_2, \ldots, x_n$.

By the construction, $S_{n+1} \subseteq S_n$ for each n.

Now, consider the "diagonal" $(f_{n,n})_{n\in\mathbb{N}}$. We claim $(f_{n,n})_{n\in\mathbb{N}}$ converges pointwise.

Indeed, for any $i\in\mathbb{N}$, $(f_{n,n}(x_i))_{n\in\mathbb{N}}$ converges because $(f_{n,n}(x_i))_{\substack{n\in\mathbb{N}\\ n\geq i}}$ converges (it is a subsequence of S_i, which converges).

This completes the proof. $\blacksquare$

Theo. [Arzella-Ascoli]

Let K be a compact metric space and $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions on K which are pointwise bounded and equicontinuous.

Then $(f_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence (on K).

Proof. As K is compact, it has a countable dense subset E.

$\left(K = \bigcup_{x\in K} B(x, 1/n).\ \text{Take finite subcover and take the union over } n \right)$

$(f_n)_{n\in\mathbb{N}}$ is pointwise bounded on E.

By the previous lemma, it has a subsequence that converges pointwise on E; let it be $(g_n)_{n\in\mathbb{N}}$.

We claim that this sequence converges uniformly on K.

$(g_i)_{i\in\mathbb{N}}$ is equicontinuous. For $\varepsilon>0$, $\exists\, \delta>0$ s.t.

$$d(x,y) < \delta \implies |g_n(x) - g_n(y)| < \varepsilon/3 \text{ for all } n.$$

Let $E = \{x_1, x_2, \ldots\}$. Then

$$K = \bigcup_{x_i\in E} B(x_i, \delta)$$

But K is compact.

So let $K = B(x_1, \delta) \cup \dots \cup B(x_m, \delta)$.

Since g_i converges pointwise, there is $n \in \mathbb{N}$ s.t.

$$|g_i(x_s) - g_j(x_s)| < \varepsilon/3 \quad \text{— (1)}$$

for all $i, j > N$

$1 \le s \le m$

Let $x \in K$. Then $x \in B(x_s, \delta)$ for some $1 \le s \le m$.

$$\Rightarrow |g_i(x) - g_i(x_s)| < \varepsilon/3 \quad \text{for all } i \quad \text{— (2)}$$

(equicontinuity)

For $i, j > N$,

$$|g_i(x) - g_j(x)| \le |g_i(x) - g_i(x_s)| + |g_i(x_s) - g_j(x_s)| + |g_j(x_s) - g_j(x)|$$

$$= \underset{(2)}{\varepsilon/3} + \underset{(1)}{\varepsilon/3} + \underset{(2)}{\varepsilon/3}$$

$\therefore (g_n)_{n \in \mathbb{N}}$ converges uniformly on K.

Corollary

Assume K is compact

Def. Let $\mathcal{A} \subseteq C(K)$. $\mathcal{A}$ is said to be an algebra if

Algebra

(i) For $f, g \in \mathcal{A}$, $f+g$, $f \cdot g$, and $c \cdot f \in \mathcal{A}$ for any $c \in \mathbb{C}$.

(complex algebra)

(if $\mathbb{R}$, then real algebra)

Def. Let $\mathcal{A}$ be an algebra in $C(K)$. $\mathcal{A}$ separates points if for any $k_1 \neq k_2 \in K$, there is $f \in \mathcal{A}$ such that $f(k_1) \neq f(k_2)$

Def. Let $\mathcal{A}$ be an algebra in $C(K)$. $\mathcal{A}$ vanishes nowhere if for any $k \in K$, there exists $f \in \mathcal{A}$ such that $f(k) \neq 0$.

Theo. [Stone-Weierstrass Theorem]

Let $\mathcal{A}$ be a real algebra in $C(K)$. If $\mathcal{A}$ separates points and vanishes nowhere, then $\overline{\mathcal{A}} = C(K)$.

(closure in $C(K)$ under sup norm)

(Weierstrass theorem is a corollary)

Theo. Let $\mathcal{A}$ be a complex algebra in $C(K)$ that is closed under conjugation, separates points, and vanishes nowhere.