Riemann Integration

Given a bounded function $f : [a,b] \to \mathbb{R}$, we define

1. A partition of $[a,b]$ as $P = \{a = x_0, x_1, \cdots, x_n = b\}$ s.t. $a < x_1 < \cdots < x_{n-1} < b$.
2. A refinement of a partition P of $[a,b]$ as another partition P' such that $P' \supseteq P$.

Given a bounded function $f : [a,b] \to R$ and a partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a,b]$, define for each i,

$$M_i = \sup\{f(t) : t \in [x_{i-1}, x_i]\} \text{ and}$$
$$m_i = \inf\{f(t) : t \in [x_{i-1}, x_i]\}.$$

Further define

$$U(P;f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \text{ and}$$
$$L(P;f) = \sum_{i=1}^{n} m_i (x_i - x_{1-1})$$

Now, observe that if P' is a partition of P, then

$$L(P;f) \le L(P';f) \le U(P';f) \le U(P;f).$$

With this motivation, we define the Riemann upper sum as

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(P;f) : P \text{ is a partition of } [a,b]\}$$

and the Riemann lower sum as

$$\underline{\int_a^b} f(x)\,dx = \sup\{L(P;f) : P \text{ is a partition of } [a,b]\}$$

f is said to be Riemann integrable if these two are equal and in this case, their value is the Riemann integral of f on $[a,b]$.

For example, consider $f(x) = x$ on $[0,1]$.
Consider $P_n = \{\frac{k}{n} : 0 \le k \le n\}$

$$U(P_n ; f) = \sum_{i=1}^{n} \left(\frac{i}{n} \times \frac{1}{n}\right) = \frac{1}{2}\left(1 + \frac{1}{n}\right)$$

$$L(P_n ; f) = \sum_{i=0}^{n-1} \left(\frac{i}{n} \times \frac{1}{n}\right) = \frac{1}{2}\left(1 - \frac{1}{n}\right)$$

$$\overline{\int_a^b} f \le \inf \{U(P_n;f) : n \in \mathbb{N}\} = \frac{1}{2}$$

$$\underline{\int_a^b} f \ge \sup \{L(P_n;f) : n \in \mathbb{N}\} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le \frac{1}{2}$$

$\Rightarrow$ f is Riemann integrable and its integrable is $\frac{1}{2}$.

Theo. If f is monotone, then f is Riemann integrable.

The proof is similar to how we evaluated $\int_0^1 x\,dx$ above.

Ex. Show the f is not integrable on $[0,1]$ where

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \cap [0,1] \\ 0, & \text{otherwise} \end{cases}$$

If f is Riemann integrable on $[a,b]$, we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Theo. Let $f:[a,b]\to\mathbb{R}$ be bounded. f is Riemann integrable iff for any $\varepsilon>0$, there is a partition P of $[a,b]$ s.t.

$$U(P;f)-L(P;f)<\varepsilon$$

Proof. Only if.

If f is Riemann int., $\exists P_1, P_2$ s.t.

$$\int_a^b f-\varepsilon/2 < L(P_1;f) \quad \text{and} \quad \int_a^b f+\varepsilon/2 > U(P_2;f)$$

Let $P=P_1\cup P_2$. Then

$$\varepsilon < U(P;f)-L(P;f)$$

If.

Let P such that $U(P;f)-L(P;f)<\varepsilon$.

Now, note that

$$L(P;f)\le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(P;f)$$

$$\Rightarrow 0\le \overline{\int_a^b} f-\underline{\int_a^b} f\le\varepsilon$$

$$\rightarrow \overline{\int_a^b} f=\underline{\int_a^b} f \quad \text{and } f \text{ is Riemann integrable.}$$

Corollary Let f be Riemann integrable on $[a,b]$ and let $P=\{x_0=a, x_1, \dots, x_n=b\}$. such that $U(P;f)-L(P;f)<\varepsilon$.

For each $i=1,2,\dots,n$, let $t_i\in[x_{i-1},x_i]$. Then

$$\left|\sum_{i=1}^n f(t_i)(x_i-x_{i-1})-\int_a^b f\right|<\varepsilon.$$

Indeed, note that

$$L(P;f)\le\sum_{i=1}^n f(t_i)(x_i-x_{i-1})\le U(P;f) \quad \text{and} \quad L(P;f)\le\int_a^b f\le U(P;f)$$

$$\Rightarrow L(P;f)-U(P;f)\le \sum_{i=1}^n f(t_i)(x_i-x_{i-1})-\int_a^b f\le U(P;f)-L(P;f)$$

Theo. If f is continuous on $[a,b]$, then it is Riemann integrable.

Proof As $[a,b]$ is compact, f is uniformly continuous (on $[a,b]$).
For $\varepsilon>0$, $\exists\,\delta>0$ s.t.

$$|x-t|<\delta \Rightarrow |f(x)-f(t)|<\varepsilon/(b-a) \quad \forall x,t\in[a,b]$$

Let $P=\{x_0=a, x_1, \dots, x_n=b\}$ with

$$\|P\| = \max_{1\le i\le n} |x_i-x_{i-1}| < \delta$$

Then f_0

$$U(P;f)-L(P;f) = \sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1})$$

$$< \sum_{i=1}^{n} \frac{\varepsilon}{(b-a)}(x_i-x_{i-1})$$

$$= \varepsilon$$

$\Rightarrow$ For any $\varepsilon>0$, there is a partition P s.t. $U(P;f)-L(P;f)<\varepsilon$.

Theo. Let f be Riemann integrable and $m\le f\le M$. Let φ be a continuous function on $[m,M]$. Then $\varphi\circ f$ is Riemann integrable.

Proof. Fix $\varepsilon>0$.
As φ is uniformly continuous on $[m,M]$,
$\exists\,\delta>0$ with $\delta<\varepsilon$ s.t. $|s-t|<\delta$

$$\Rightarrow |\varphi(s)-\varphi(t)|<\varepsilon \quad \forall s,t\in[m,M]$$

As f is Riemann integrable, there exists $P=\{a=x_0,x_1,\dots,x_n=b\}$
s.t. $U(P;f)-L(P;f)<\delta^2$.

Let $M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$ $\quad m_i = \inf_{x\in[x_{i-1},x_i]} f(x)$

$$M_i^* = \sup_{x\in[x_{i-1},x_i]} (\varphi\circ f)(x) \qquad m_i^* = \sup_{x\in[x_{i-1},x_i]} (\varphi\circ f)(x)$$

$A=\{i\in\{1,\dots,n\} : M_i-m_i<\delta\}$
$B=\{i\in\{1,\dots,n\} : M_i-m_i\ge\delta\}$

Observe that
$$\left.\begin{array}{l} i\in A \Rightarrow M_i^* - m_i^* < \varepsilon \\ i\in B \Rightarrow M_i^* - m_i^* < 2k \end{array}\right\} \text{---} \text{(ii)}$$

where $k = \sup_{t\in[m,M]} |\varphi(t)|$

$$\therefore\ \delta \sum_{i\in B} (x_i - x_{i-1}) \le \sum_{i\in B} (M_i - m_{i-1})(x_i - x_{i-1})$$

$$\le U(P;f) - L(P;f) < \delta^2$$

$$\Rightarrow \sum_{i\in B} (x_i - x_{i-1}) < \delta \text{ --- (i)}$$

Finally,

$$U(P;\varphi\circ f) - L(P;\varphi\circ f)$$
$$= \sum_{i=1}^{n} (M_i^* - m_i^*)(x_i - x_{i-1})$$
$$= \sum_{i\in A} (M_i^* - m_i^*)(x_i - x_{i-1}) + \sum_{i\in B} (M_i^* - m_i^*)(x_i - x_{i-1})$$
$$< \varepsilon(b-a) + 2k\delta \quad (\text{by (i) and (ii)})$$
$$< \varepsilon(b-a+2k)$$

Changing ε to $\frac{\varepsilon}{b-a+2k}$ yields the result.

Theo. If f_1 and f_2 are Riemann integrable then f_1+f_2 is Riemann integrable and further,

$$\int_a^b (f_1+f_2) = \int_a^b f_1 + \int_a^b f_2$$

Proof. Integrability follows from the fact that for any partition P of $[a,b]$,

$$L(P;f_1) + L(P;f_2) \le L(P;f_1+f_2) \le U(P;f_1+f_2) \le U(P;f_1) + U(P;f_2)$$

Further, there exists partition P for any $\varepsilon>0$ s.t.

$$\int_a^b f_1+f_2 \le U(P;f_1+f_2) \le U(P;f_1)+U(P;f_2) \le \int_a^b f_1 + \int_a^b f_2 + \varepsilon$$

$$\Rightarrow \int_a^b f_1+f_2 \le \int_a^b f_1 + \int_a^b f_2$$

Replacing f_1 with $-f_1$ and f_2 with $-f_2$, we can infer the result.

1. For Riemann integrable f and $\alpha \in \mathbb{R}$,
$$\int \alpha f = \alpha \int f \qquad (x \mapsto \alpha x \text{ is continuous})$$

2. Suppose f is Riemann integrable on $[a,c]$ and $[c,b]$ where $a<c<b$.
Then f is Riemann integrable on $[a,b]$ and
$$\int_a^b f = \int_a^c f + \int_c^b f \qquad (\text{Split any partition of } [a,b] \text{ at } c)$$

3. If f and g are Riemann integrable with $f \leq g$, then
$$\int f \leq \int g \qquad (\text{Just compare upper/lower limits})$$

4. Let $f \geq 0$ be Riemann integrable such that f is continuous at x_0 and $f(x_0) > 0$. Then
$$\int f > 0 \qquad \left(\begin{array}{l}\text{There is some interval around} \\ x_0 \text{ where } f \text{ is positive}\end{array}\right)$$

5. For Riemann integrable f, $|f|$ is Riemann integrable and further,
$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx \qquad (-|f| \leq f \leq |f|).$$

Theo.

Let f be Riemann integrable on $[a,b]$. Then for $a \leq x \leq b$, define $F(x) = \int_a^x f(t)\,dt$.

F is continuous on $[a,b]$. Further, if f is continuous at x, then F is differentiable at x with $F'(x) = f(x)$

Proof. Let $x, y \in [a,b]$ with $x > y$.
$$F(x) - F(y) = \int_a^x f(t)\,dt - \int_a^y f(t)\,dt = \int_y^x f(t)\,dt$$
$$|F(x) - F(y)| \leq \int_y^x f(t)\,dt \leq M(x-y) \qquad (\text{as } f \text{ is bounded})$$

Thus F is continuous on $[a,b]$.

Now suppose f is continuous at x, that is, for $\varepsilon>0$, $\exists\,\delta>0$

$$\underset{t\in[a,b]}{|t-x|<\delta} \Rightarrow |f(t)-f(x)|<\varepsilon$$

For $x-\delta<t<x+\delta$,

$$\left|\frac{F(t)-F(x)}{t-x}-f(x)\right| = \left|\frac{1}{t-x}\int_x^t f(y)\,dy - f(x)\right|$$

$$= \left|\frac{1}{t-x}\left[\int_x^t (f(y)-f(x))\,dy\right]\right|$$

$$\le \left|\frac{1}{t-x}\int_x^t \varepsilon\,dy\right| \quad (\text{as } |y-x|\le|t-x|<\delta)$$

$$= \varepsilon$$

$\Rightarrow$ F is differentiable at x and $F'(x)=f(x)$.

Theo.

Let f be Riemann integrable on $[a,b]$. Let there be a differentiable function F on $[a,b]$ s.t. $F'(x)=f(x)$ $\forall x\in[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b)-F(a).$$

Proof. Fix $\varepsilon>0$. Let there be a partition $P=\{a=x_0,x_1,\dots,x_n=b\}$ of $[a,b]$ s.t. $U(P;f)-L(P;f)<\varepsilon$.

Now, by the Mean Value Theorem, for $i=1,2,\dots,n$

$$F(x_i)-F(x_{i-1}) = (x_i-x_{i-1})f(t_i) \quad \text{for some } t_i\in(x_{i-1},x_i)$$

As we have shown earlier,

$$\left|\sum_{i=1}^n (x_i-x_{i-1})f(t_i) - \int_a^b f(x)\,dx\right| < \varepsilon$$

$$\Rightarrow \left|\sum_{i=1}^n F(x_i)-F(x_{i-1}) - \int_a^b f(x)\,dx\right| < \varepsilon$$

$$\Rightarrow \left|(F(b)-F(a)) - \int_a^b f(x)\,dx\right| < \varepsilon$$

$$\Rightarrow \int_a^b f(x)\,dx = F(b)-F(a)$$

Theo. Let F and G be differentiable on $[a,b]$ such that $F' = f$ and $G' = g$ are Riemann integrable.

Then $$\int_a^b F(x)\, G(x)\, dx = F(b)\,G(b) - F(a)\,G(a) - \int_a^b f(x)\, G(x)\, dx$$

Let $A \subseteq \mathbb{R}$. A is said to be of measure zero if for any $\varepsilon > 0$, there exists a sequence of open intervals $(I_n)_{n\in\mathbb{N}}$ s.t
$$A \subseteq \bigcup_{n=1}^{\infty} I_n \quad \text{and} \quad \sum_{n=1}^{\infty} \ell(I_n) < \varepsilon.$$
where $\ell((a,b)) = b-a$

For example, let $A = \{x_1, x_2, \cdots\}$ be a countable set.
Fix $\varepsilon > 0$. For each n, let $I_n = \left(x_n - \frac{\varepsilon}{2^{n+2}} ,\ x_n + \frac{\varepsilon}{2^{n+2}} \right)$

Then $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} \ell(I_n) = \varepsilon/2 < \varepsilon.$

We see that any countable set is of measure zero.

Ex. If $A \subseteq B$ and B is of measure zero, show that A is of measure zero.
Ex. Show that a countable union of measure zero sets is of measure zero.

Def. Let $f : E(\subseteq \mathbb{R}) \to \mathbb{R}$ be bounded. Let $B \subseteq E$.
We define the oscillation of f at B by
$$\begin{aligned} \omega_f(B) &= \operatorname{diam} f(B) \\ &= \sup_{x\in B} f(x) - \inf_{x\in B} f(x) \end{aligned}$$

We define the oscillation of f at $x \in E$ by
$$\omega_f(x) = \inf_{\delta > 0} \omega_f(B(x,\delta))$$

We see that a function f is continuous at x iff $\omega_f(x) = 0$.
(Show this)

Def. Let f be a function. We define Δ_f as the set of discontinuities of f.

Theo. Let $f: [a,b] \to \mathbb{R}$ be bounded. f is Riemann integrable if and only if Δ_f is of measure zero.

Proof. Suppose Δ_f is of measure zero. Let $\varepsilon > 0$.

- There is a sequence of open intervals $(I_n)_{n\in\mathbb{N}}$ s.t. $\Delta_f \subseteq \bigcup I_n$ and $\sum l(I_n) < \varepsilon$.

  Let $J = \{ n \in \mathbb{N} : \omega_f(I_n) > \varepsilon \}$

  $$V_\varepsilon = \bigcup_{n\in J} I_n$$

  Clearly, $\sum_{n\in J} l(I_n) \leq \sum_{n\in\mathbb{N}} l(I_n) < \varepsilon$

  For some $N \in \mathbb{N}$, let $x_i = a + (b-a)(i/N)$.
  We then have a partition $\{a = x_0, x_1, \dots, x_N = b\}$

- We shall show that there exists $N \in \mathbb{N}$ such that for any $i \in \{1, 2, \dots, N\}$, if $\omega_f([x_{i-1}, x_i]) > \varepsilon$, then $[x_{i-1}, x_i] \subseteq V_\varepsilon$.

  Suppose otherwise.
  Then for every $N \in \mathbb{N}$, $\exists\, i \in \{1, 2, \dots, N\}$ s.t.
  $$\omega_f([x_{i-1}, x_i]) > \varepsilon \quad \text{and} \quad [x_{i-1}, x_i] \cap V_\varepsilon^c \neq \emptyset.$$
  $\Rightarrow \exists\, s_N, t_N, z_N \in [x_{i-1}, x_i]$ s.t.
  $$\omega_f([x_{i-1}, x_i]) \geq f(s_N) - f(t_N) > \varepsilon$$
  $$\text{and } z_N \in V_\varepsilon^c.$$
  As $(s_N)_{N\in\mathbb{N}}$ is a sequence in $[a,b]$, it has a convergent subsequence $(s_{N_k})_{k\in\mathbb{N}}$. Let $s_{N_k} \to y$.
  $$|s_N - t_N| \leq \frac{b-a}{N} \quad \text{and} \quad |s_N - z_N| \leq \frac{b-a}{N}$$
  $$\Rightarrow t_{N_k} \to y \text{ and } z_{N_k} \to y$$

  However, $f(s_{N_k}) - f(t_{N_k}) > \varepsilon \Rightarrow f$ is discontinuous at y.

$$\Rightarrow y \in \Delta_f \subseteq \bigcup_{n=1}^{\infty} I_n$$

$\Rightarrow y \in I_j$ for some $j \in \mathbb{N}$.

Now, note that V_ε^c is closed and $(z_{N_k})_{k\in\mathbb{N}}$ is in V_ε^c.
That is, $y \notin V_\varepsilon$.
$\Rightarrow \omega_f(I_j) \le \varepsilon$.
As I_j is open and $y \in I_j$,
(s_{N_k}) and (t_{N_k}) are in I_j for sufficiently large k.
As $\omega_f(I_j) \le \varepsilon$,
$f(s_{N_k}) - f(t_{N_k}) \le \varepsilon$ for sufficiently large k.
This is a contradiction!
(Our basis for choosing s_N and t_N was $f(s_N) - f(t_N) > \varepsilon$)
This proves our claim.

- Fix $N \in \mathbb{N}$ as obtained in the previous point.
  Let $P = \{a = x_0, x_1, \cdots, x_N = b\}$.

$$U(P;f) - L(P;f) = \sum_{i=1}^{N} \frac{b-a}{N}\,\omega_f([x_{i-1}, x_i])$$

$$\le k \cdot \frac{b-a}{N}\,\omega_f([a,b])$$

$$+ \sum_{\substack{1 \le i \le N \\ i \notin J}} \frac{b-a}{N}\,\omega_f([x_{i-1}, x_i])$$

where $k = \left|\{ i : [x_{i-1}, x_i] \subseteq V_\varepsilon \}\right|$

$$\le \varepsilon\,\omega_f([a,b]) + \varepsilon \cdot (b-a)$$

$\left(k \cdot \frac{b-a}{N} < \varepsilon \text{ as } \ell(V_\varepsilon) < \varepsilon \right)$

$$= \varepsilon\,(b-a + \omega_f([a,b]))$$

Changing ε to $\dfrac{\varepsilon}{b-a+\omega_f([a,b])}$ proves the result.

Conversely, suppose f is Riemann integrable on $[a,b]$. We must show that Δ_f is of measure zero.

$$\Delta_f = \{x \in [a,b] : \omega_f(x) > 0\}$$
$$= \bigcup_{k=1}^{\infty} \{x \in [a,b] : \omega_f(x) > 1/k\}$$

We shall prove that each of $\{x \in [a,b] : \omega_f(x) > 1/k\}$ is of measure zero.
For $\varepsilon > 0$, there is a partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a,b]$ s.t. $U(P;f) - L(P;f) < \varepsilon/2k$

$$\Rightarrow \sum_{i=1}^{n} \omega_f([x_{i-1}, x_i])(x_i - x_{i-1}) < \varepsilon/2k$$

Let $F = \{i \in [n] : (x_{i-1}, x_i) \cap \{x : \omega_f(x) > 1/k\} \neq \emptyset\}$
If $i \in F$, then $\omega_f([x_{i-1}, x_i]) > 1/k$

Then consider

$$\frac{1}{k} \sum_{i \in F} (x_i - x_{i-1}) < \sum_{i \in F} \omega_f([x_i, x_{i-1}])(x_i - x_{i-1})$$
$$\leq \sum_{i=1}^{n} \omega_f([x_i, x_{i-1}])(x_i - x_{i-1}) < \varepsilon/2k$$

$$\Rightarrow \sum_{i \in F} (x_i - x_{i-1}) < \varepsilon/2 .$$

Δ_f then has cover $\{(x_{i-1}, x_i) : i \in F\} \cup \underbrace{\{\{x_i\} : 1 \leq i \leq n\}}_{\text{There is an open cover of length } < \varepsilon/2 \text{ (it is of measure zero)}}$.

The resulting open cover then has length $< \varepsilon$ and thus, $\{x \in [a,b] : \omega_f(x) > 1/k\}$ is of measure zero and so is Δ_f.