Functions

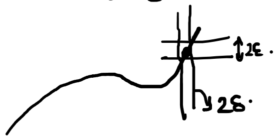
Def. Let $f: E \to Y$ be a function where (X, d_X), (Y, d_Y) are metric spaces with $E \subseteq X$ and $p \in E'$. Then

$$\lim_{x \to p} f(x) = q$$

if for each $\varepsilon > 0$, $\exists \delta > 0$ such that

$$0 < d_X(x, p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon$$
$$\text{and } x \in E$$

The closer x gets to p, the closer $f(x)$ gets to $\lim_{y \to p} f(y)$.



For example, if $f(x) = x^2$ on $\mathbb{R}$, then $|x| < \varepsilon^{1/2} \Rightarrow f(x) < \varepsilon$ for any $\varepsilon > 0$. This implies $\lim_{x \to 0} f(x) = 0$.
Note that $p \in E'$.

Def. If $f: \mathbb{R} \to \mathbb{R}$, we write $\lim_{x \to \infty} f(x) = \alpha$ if for any $\varepsilon > 0$, $\exists M > 0$ such that

Limit

$$x > M \Rightarrow |f(x) - \alpha| < \varepsilon.$$

We can also define the limit of f sequentially as:

Theo. Let $f: E \to Y$ be a function where (X, d_X), (Y, d_Y) are metric spaces with $E \subseteq X$ and $p \in E'$. Then

$$\lim_{x \to p} f(x) = q$$

iff for any sequence $(p_n)_{n \in \mathbb{N}}$ in E such that $p_n \neq p$ and $(p_n)_{n \in \mathbb{N}} \to q$, $(f(p_n))_{n \in \mathbb{N}} \to q$.

Let $(p_n)_{n \in \mathbb{N}}$ satisfy the given conditions. Then $\lim_{x \to p} f(x) = q$ iff for any $\varepsilon > 0$, $\exists \delta > 0$ s.t. $0 < d_X(x, p) < \varepsilon \Rightarrow d_Y(f(x), q) < \varepsilon$.
Then if $p_n \to p$, $\exists n_0 \in \mathbb{N}$ s.t. $0 < d_X(p_n, p) < \delta$ for all $n > n_0$

$$\Rightarrow 0 < d_Y(f(p_n), q) < \varepsilon \text{ for all } n > n_0 \Rightarrow (f(p_n)) \to q.$$

Conversely, let $f(p_n) \to q$ for all such sequences $(p_n)_{n\in\mathbb{N}}$.
If $\lim_{x\to p} f(x) \neq q$, then $\exists \varepsilon_0 > 0$ such that for all $\delta > 0$ s.t. there exists $x \in E$ with $0 < d_X(x,p) < \delta$ such that $d_Y(f(x), q) \geq \varepsilon_0$.
(the negation of the statement)

Let x_n be such a choice of x for $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$.

Then $x_n \to p$. However, $f(x_n)$ does not converge to q. This is a contradiction, and therefore $\lim_{x\to p} f(x) = q$.

Ex. Show that $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Think of two sequences that converge to 0 but have different functional limits.

Let $f: E \to Y$ and $g: E \to Y$. Then

1. $\lim_{x\to p} (f+g)(x) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x)$
2. $\lim_{x\to p} (fg)(x) = \lim_{x\to p} f(x) \cdot \lim_{x\to p} g(x)$
3. $\lim_{x\to p} \left(\frac{f}{g}\right)(x) = \dfrac{\lim_{x\to p} f(x)}{\lim_{x\to p} g(x)}$ if both sides are well-defined.

These are easy to prove using the corresponding results for sequences.

Def. Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces with $E \subseteq X$. $f: E \to Y$ is said to be continuous at a point $p \in E$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in E \text{ and } d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon.$$

The closer x gets to p, the closer $f(x)$ gets to $f(p)$.
Note that $p \in E$ here (not E' like in the definition of a limit)

Theo. Suppose $f: E \to Y$ and $p \in E$. If p is not a limit point of E, then f is continuous at p.

Proof: If p is not a limit point of E, $\exists \delta_0 > 0$ s.t. $E \cap B_X(p, \delta_0) = \{p\}$.
Then for any $\varepsilon > 0$, $\underset{\substack{\Downarrow \\ x = p}}{d_X(x,p) < \delta_0} \Rightarrow \underset{\substack{\downarrow \\ =0}}{d_Y(f(x), f(p))} < \varepsilon$

For example, the function $f: \{1\} \to \mathbb{R}$ where $f(1) = 0$ is continuous at $\{1\}$.
Note that the limit is not defined; it only exists if $p \in E'$, which is exactly not the case here.

Theo. Let (X, d_X) and (Y, d_Y) be metric spaces with $E \subseteq X$ and $f: E \to Y$. f is continuous at $p \in E$ iff for any $(p_n)_{n \in \mathbb{N}}$ in E, $p_n \to p$ implies $f(p_n) \to f(p)$.

Note that here, p_n can be equal to p.
The proof is very similar to the earlier one (for the limit) so we omit it.

Theo. Let $f: E \to Y$. If $p \in E' \cap E$, then f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$.

This follows directly from the definition.

Let us look at continuity on $\mathbb{R}$ now. Let $f: \mathbb{R} \to \mathbb{R}$. f is continuous at p if given $\varepsilon > 0$, $\exists \delta > 0$ such that

$$\underset{\substack{\downarrow \\ \{f(x) : x \in (p-\delta, p+\delta)\}}}{f((p-\delta, p+\delta))} \subseteq (f(p) - \varepsilon, f(p) + \varepsilon)$$

$$\left[f(B_X(p, \delta)) \subseteq \underset{\text{generally}}{B_Y(f(p), \varepsilon)} \right]$$

So for example, $f \begin{cases} x(x-1), & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ is continuous only at $\{0, 1\}$

Theo. Let $(X, d_X), (Y, d_Y)$ and (Z, d_Z) be metric spaces with $E \subseteq X$. If we have
If $f: E \to Y$ and $g: f(E) \to Z$ are continuous at p and $f(p)$ for some $p \in E$. Then $g \circ f: E \to Z$ is continuous at p.

This is obvious using the sequential criterion for continuity.

Let $f, g: X \to \mathbb{R}$ be continuous at p. Then $(f \pm g)$, (fg), and (f/g) are continuous at p.
(if $g(x) \neq 0$ $\forall x \in X$)
This can be proved using the above theorem.

Theo. Let $f: X \to Y$ given metric spaces X and Y. f is continuous on X if and only if $f^{-1}(V)$ is open for every open $V \subseteq X$.
$\hookrightarrow \{u \in X : f(u) \in V\}$

Proof Let f be continuous on X and V be open in Y. Let $x \in f^{-1}(V)$. Since $f(x) \in V$, there exists $\varepsilon > 0$ s.t. $B_Y(f(x), \varepsilon) \subseteq V$. As f is continuous, there exists $\delta > 0$ s.t. $y \in B_X(x, \delta) \Rightarrow f(y) \in B_Y(f(x), \delta)$. This implies $f(B_X(x, \delta)) \subseteq B_Y(f(x), \delta) \subseteq V$, that is, $B_X(x, \delta) \subseteq f^{-1}(V)$ for some $\delta > 0$. This just says that $f^{-1}(V)$ is open.

Conversely, suppose $f^{-1}(V)$ is open in X for every open $V \subseteq Y$.
Let $x \in X$ and $\varepsilon > 0$. We must find a $\delta > 0$ s.t. $f(B_X(x, \delta)) \subseteq B_Y(f(x), \varepsilon)$
As $B_Y(f(x), \varepsilon)$ is open, $f^{-1}(B_Y(f(x), \varepsilon))$ is open. As x belongs to this set, the result follows. ■

Corollary $f: X \to Y$ is continuous on X if and only if $f^{-1}(V)$ is closed for every closed $V \subseteq Y$.

This is easy to show using the fact that $f^{-1}(V^c) = (f^{-1}(V))^c$.

Ex: Let $GL(n,\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : \det A \neq 0\}$ and $SL(n,\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : \det A = 1\}$. Show that $GL(n,\mathbb{R})$ and $SL(n,\mathbb{R})$ are open and closed (but not compact) respectively in $M_{n\times n}(\mathbb{R})$, the set of $n\times n$ matrices in $\mathbb{R}$ with the distance defined by the corresponding distance in $\mathbb{R}^{n^2}$. Further show that $GL(n,\mathbb{R})$ is disconnected.

Def. Let $f: X \to \mathbb{R}^n$ where X is a metric space. f is said to be **bounded** if there exists some $M>0$ s.t. $d(f(x),0) \le M$ for all $x \in X$.

Bounded

$d(f(x),0)$ often written as $|f(x)|$

Theo. Let X be a compact metric space and Y be a metric space. If $f: X \to Y$ is compact, then $f(X)$ is compact.

(Continuous image of a compact set is compact)

Proof. Let $(V_\alpha)_{\alpha \in A}$ be an open cover of $f(X)$. That is,

$$X \subseteq f^{-1}\Big(\bigcup_{\alpha \in A} V_\alpha\Big) = \bigcup_{\alpha\in A} f^{-1}(V_\alpha)$$

why?

$\Rightarrow (f^{-1}(V_\alpha))_{\alpha \in A}$ is an open cover of X. As X is compact,

$$X = f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n}) \text{ for some } \alpha_1, \dots, \alpha_n \in A$$

$$\begin{aligned} \Rightarrow f(X) &= f(f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n})) \\ &\overset{\text{why?}}{=} f(f^{-1}(V_{\alpha_1})) \cup \cdots \cup f(f^{-1}(V_{\alpha_n})) \\ &\subseteq V_{\alpha_1} \cup \cdots \cup V_{\alpha_n} \end{aligned}$$

$f(\{x : f(x) \in V_\alpha\}) \subseteq V_\alpha$

$\blacksquare$

Theo. Let $f: X \to \mathbb{R}$ be continuous and X be compact. Then there exist $p, q \in X$ such that $f(p) = \sup_{x \in X} f(x)$ and $f(q) = \sup_{x \in X} f(x)$.

(f attains its supremum and infimum)

This is obvious as $f(X)$ is compact and thus closed and bounded. Boundedness implies existence of sup/inf and closedness implies that it is in $f(X)$.

Lemma. Let $f: X \to Y$ be continuous and X be connected. Then $f(X)$ is connected.

Proof. Suppose $f(X)$ is disconnected. Then there exist open $U, V \subseteq Y$ with $U \cap V = \emptyset$ such that $f(X) = U \cup V$.

Then since $\underbrace{X = f^{-1}(f(X))}_{(\text{why?})} = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$ which are disjoint open sets (as f is continuous), which is a contradiction.

Theo. [Intermediate Value Theorem]

Intermediate Value Theorem

Let $f: [a,b] \to \mathbb{R}$ be continuous such that $f(a) < f(b)$. Given c such that $f(a) < c < f(b)$, there exists $p \in [a,b]$ such that $f(p) = c$.

Proof. As $[a,b]$ is connected and compact, $f([a,b])$ is connected and compact. That is, $f([a,b])$ is a closed and bounded interval.

$\Rightarrow [f(a), f(b)] \subset f([a,b])$. The result follows.

Corollary. If $f[a,b] \to \mathbb{R}$ is continuous with $f(a) < 0$ and $f(b) > 0$, there exists $c \in [a,b]$ such that $f(c) = 0$.

Def. Let $f: X \to Y$ where X, Y are metric spaces. We say f is uniform continuous on X if for every $\varepsilon > 0$, $\exists \delta > 0$ s.t. for all $x, y \in X$,

Uniform Continuity

$$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

(This differs from continuity because δ is independent of x and y)

In continuity, while each $\delta_x > 0$, we do not know if $(\inf \delta_x) > 0$.

For example, if $f:[0,1]\to\mathbb{R}$ is given by $f(x)=x^2$, then given any $\varepsilon>0$, choose $\delta=\varepsilon/2$.

Ex. Show that $f:(0,1)\to\mathbb{R}$ given by $f(x)=\frac{1}{x}$ is not uniform continuous.

Consider $|f(x)-f(1/2)|$.

Theo. Let $f:X\to Y$ be continuous. If X is compact, then f is uniformly continuous.

Proof. For each $p\in X$ and $\varepsilon>0$, there exists $\delta(p)>0$ s.t.

$$f(B_x(p,\delta(p)))\subseteq B_y(f(p),\varepsilon/2)$$

Consider $\{B_x(p,\delta(p)/2)\}_{p\in X}$ be an open cover of X.

As X is compact, let there be $p_1,\dots,p_n$ s.t.

$$X=\bigcup_{i=1}^{n} B_x\left(p_i,\delta(p_i)/2\right)$$

Let $\delta=\min\limits_{1\le i\le n}\left(\frac{\delta(p_i)}{2}\right)>0.$

We claim that for any $x,y\in X$,

$$d_x(x,y)<\delta \Rightarrow d_y(f(x),f(y))<\varepsilon.$$

Indeed, given any $x\in X$, $x\in B_x\left(p_m,\frac{\delta(p_m)}{2}\right)$ for some $1\le m\le n$.

Then $d_x(x,y)<\delta \Rightarrow d_x(y,p_m)\le d_x(p_m,x)+d_x(x,y)$

$$<\delta(p_m)$$

As $d_y(f(x),f(p_m))<\varepsilon/2$ and $d(f(p_m),f(y))<\varepsilon/2$,

$d_y(f(x),f(y))<\varepsilon$

■

Def. Let $f:(a,b)\to\mathbb{R}$. Then for any $x\in[a,b)$

Limit on one side

$$f(x^+)=\lim_{t\to x^+} f(t)=q \text{ if for any } \varepsilon>0,$$

there exists $\delta>0$ s.t.

$$x<t<x+\delta \Rightarrow |f(t)-q|<\varepsilon.$$

Equivalently, if $(t_n)_{n\in\mathbb{N}}$ is a sequence in (x,b) such that $t_n\to x$, then $f(t_n)\to q$.

Similarly, for any $x\in(a,b]$,

$$f(x^-)=\lim_{t\to x^-} f(t)=q \text{ if for any } \varepsilon>0,$$

there exists $\delta>0$ s.t.

$$x-\delta<t<x \Rightarrow |f(t)-q|<\varepsilon.$$

Equivalently, if $(t_n)_{n\in\mathbb{N}}$ is a sequence in (a,x) such that $t_n\to x$, then $f(t_n)\to q$.

Clearly, if $f:\mathbb{R}\to\mathbb{R}$ is continuous at p, then

Types of Discontinuities

$$f(p)=\lim_{t\to p} f(t)=f(t^+)=f(t^-)$$

Let $f:(a,b)\to\mathbb{R}$ be discontinuous at $p\in(a,b)$. Then either

(a) $\lim_{t\to p^+} f(t)$ or $\lim_{t\to p^-} f(t)$ do not exist,

(b) they exist and $\lim_{t\to p^+} f(t)\neq\lim_{t\to p^-} f(t)$, or

(c) $\lim_{t\to p^+} f(t)=\lim_{t\to p^-} f(t)\neq f(p)$.

If f satisfies (a), then the discontinuity at p is called a second kind discontinuity.

Eg. $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

If f satisfies (b), then the discontinuity at p is called a first kind discontinuity or a jump discontinuity.

Eg. $f(x) = \begin{cases} x/|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$

If f satisfies (c), then the discontinuity at p is called a removable discontinuity.

Eg. $f(x) = \begin{cases} |x|, & x \neq 0 \\ 1, & x = 0 \end{cases}$ (the discontinuity can be "removed" by changing $f(p)$.)

Theo. If f is a monotone function, then it only has discontinuities of the first kind. Further, the number of discontinuities is atmost countable.

Proof. Suppose that $f: [a,b] \to \mathbb{R}$ is monotone increasing.

For any $x \in [a,b)$, $f(x^+) = \inf\{f(t) : x < t \leq b\}$

$y \in (a,b]$, $f(y^-) = \sup\{f(t) : a \leq t < y\}$

this follows due to the monotone nature of f.

$f(x^+)$ exists as $\{f(t) : x < t \leq b\}$ is lower-bounded by $f(x)$. $f(y^-)$ exists similarly. ($\Rightarrow$ discontinuities of the second kind cannot occur)

Further, for any $x \in (a,b)$, we have $f(x^-) \leq f(x) \leq f(x^+)$.

($\Rightarrow$ removable discontinuities cannot occur)

Now, let x be a point of discontinuity of f. Then $f(x^-) < f(x^+)$.

Choose $r(x) \in \mathbb{Q}$ such that $f(x^-) < r(x) < f(x^+)$.

$r : E \to \mathbb{Q}$ is one-one. $(x_1 < x_2 \Rightarrow r(x_1) < r(x_2))$

set of discontinuities.

As $r(E)$ is countable, E must be countable.

(subset of $\mathbb{Q}$)

Differentiability

Def. Let $f:[a,b]\to\mathbb{R}$. f is said to be differentiable at $x\in(a,b)$ if

Differentiability $\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$ exists.

f is differentiable at a if $\lim_{t\to a^+}\frac{f(t)-f(a)}{t-a}$ exists.

f is differentiable at b if $\lim_{t\to b^-}\frac{f(t)-f(b)}{t-b}$ exists.

(There is no similar notion for general metric spaces)

The value of this limit is denoted $f'(x)$.

Theo. If f is differentiable at x, then

$$\lim_{t\to x}(f(t)-f(x)) = \underbrace{\lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}\right)\lim_{t\to x}(t-x)}_{\text{As both limits exist.}} = 0$$

$\Rightarrow f$ is continuous at x.

Def. Let (X,d) be a metric space and $f:X\to\mathbb{R}$. We say f has a

- local maximum at p if $\exists\,\delta>0$ s.t.

local maximum $f(x)\le f(p)$ for all $x\in B_X(p,\delta)$.

- local minimum at p if $\exists\,\delta>0$ s.t.

local minimum $f(x)\ge f(p)$ for all $x\in B_X(p,\delta)$.

Theo. Let $f:[a,b]\to\mathbb{R}$ be a function with local maximum/minimum at $x\in(a,b)$. If f is differentiable at x, then $f'(x)=0$.

Proof Suppose local maximum. $\exists\,\delta>0$ s.t. $x-\delta<y<x+\delta \Rightarrow f(y)\le f(x)$

For $x<t<x+\delta$, $\frac{f(t)-f(x)}{t-x}\le 0 \Rightarrow \lim_{t\to x^+}\frac{f(t)-f(x)}{t-x}\le 0 \Rightarrow f'(x)\le 0$.

Similarly, we have $f'(x)\ge 0$. $\Rightarrow f'(x)=0$.
(same thing on $x-\delta<t<x$)

Theo. Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then
$\exists x \in (a,b)$ s.t.
$$(f(b) - f(a))\, g'(x) = (g(b) - g(a))\, f'(x)$$

Proof. Define $h : [a,b] \to \mathbb{R}$ by
$$h(t) = (f(b) - f(a))\, g(t) - (g(b) - g(a))\, f(t)$$

Then
$$h(a) = h(b) = f(b)g(b) - g(b)f(a).$$
h is continuous on $[a,b]$ and differentiable on (a,b).
We claim $h'(x) = 0$ for some $x \in (a,b)$.
If h is constant, we are done.
Otherwise,

a. If $\exists t \in (a,b)$ s.t. $h(t) > h(a)$.
$\Rightarrow \exists x \in [a,b]$, $h(x) = \sup\limits_{y \in [a,b]} h(y)$ (as h is continuous on compact set $[a,b]$)
Further, $x \neq a$ and $x \neq b$ (why?)
$\Rightarrow h$ has a local maximum at x, that is, $h'(x) = 0$.

b. Similarly, if $\exists t \in (a,b)$ s.t. $h(t) < h(a)$, h has a local minimum at some $x \in (a,b)$.

This completes the proof. $\square$

Corollary. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then
$\exists x \in (a,b)$ s.t.
$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Corollary Let $f:[a,b]\to\mathbb{R}$.

1. If $f'(x)\geq 0$ on (a,b), then f is monotonically increasing.
2. If $f'(x)=0$ on (a,b), then f is constant.
3. If $f'(x)\leq 0$ on (a,b), then f is monotonically decreasing.

Theo. [Taylor's Theorem]

Taylor's Theorem

Suppose $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(n-1)}$ is continuous on $[a,b]$ and differentiable on (a,b).

Then for $\alpha<\beta$ in $[a,b]$, there exists $x\in(\alpha,\beta)$ s.t.

$$f(\beta)=f(\alpha)+(\beta-\alpha)f^{(1)}(\alpha)+\frac{(\beta-\alpha)^2}{2!}\cdot f^{(2)}(\alpha)$$

$$+\cdots+\frac{(\beta-\alpha)^{n-1}}{(n-1)!}f^{(n-1)}(\alpha)+\frac{(\beta-\alpha)^n}{n!}f^{(n)}(x).$$

Proof. Let

$$p(t)=f(\alpha)+(t-\alpha)f^{(1)}(\alpha)+\cdots+\frac{(t-\alpha)^{n-1}}{(n-1)!}f^{(n-1)}(\alpha).$$

Define M by

$$f(\beta)=p(\beta)+M(\beta-\alpha)^n$$

Define g by

$$g(t)=f(t)-p(t)-M(t-\alpha)^n$$

We shall show that $M=\dfrac{f^{(n)}(x)}{n!}$ for some $x\in(a,b)$

We have

$$g^{(n)}(t)=f^{(n)}(t)-n!\,M \quad \text{for any } t\in(a,b)$$

$\Rightarrow$ We shall show that $\underline{g^{(n)}(x)=0}$ for some $x\in(a,b)$.

As $g(\alpha)=g(\beta)=0$,

$\exists\, x_1\in(\alpha,\beta)$ s.t. $g^{(1)}(x_1)=0$. Also, $g^{(1)}(\alpha)=0$.

$\Rightarrow \exists\, x_2\in(\alpha,x_1)$ s.t. $g^{(2)}(x_2)=0$. $g^{(2)}(\alpha)=0$.

$\vdots$

$\Rightarrow \exists\, x_n\in(\alpha,x_{n-1})$ s.t. $g^{(n)}(x_n)=0$. This completes the proof. ∎

$(x_n\in(\alpha,\beta))$

Theo. [Intermediate Value Theorem for differentiation]

Intermediate Value for Differentiation

Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose $f'(a)<\lambda<f'(b)$. Then $\exists x\in(a,b)$ s.t. $f'(x)=\lambda$.

Corollary. f' cannot have discontinuities of the first kind.

Proof. Define $g(t)=f(t)-\lambda t$.

We must show $\exists x\in(a,b)$ $g'(x)=0$.

g is continuous on $[a,b]$ and differentiable on (a,b).

$\Rightarrow g$ attains its infimum at some point $y\in[a,b]$.

If $y=a$, then $g(a)\le g(t)\quad \forall t\in[a,b]$

$$\Rightarrow g'(a)\ge 0$$
$$\Rightarrow f'(a)\ge\lambda \quad\Rightarrow \text{Contradiction.}$$

If $y=b$, then $g(b)\le g(t)\quad\forall t\in[a,b]$

$$\Rightarrow g'(b)\le 0$$
$$\Rightarrow f'(b)\le\lambda \quad\Rightarrow \text{Contradiction.}$$

$\therefore y\in(a,b)$. As y is a local minimum, $g'(y)=0$

$$\Rightarrow f'(y)=\lambda.$$

There is also a generalized version of the MVT:

Theo. Let $f:[a,b]\to\mathbb{R}^n$ be continuous on $[a,b]$ and differentiable on (a,b). Then $\exists x\in(a,b)$ such that

$$\|f'(b)-f'(a)\|\le(b-a)\,\|f'(x)\|$$

(Use MVT on $\phi:[a,b]\to\mathbb{R}$ given by $\phi(t)=\langle f(b)-f(a),f(t)\rangle$)