# Logic and Propositional Calculus.

**Def.** Propositions are statements that are either true or false (but not both).

A proposition could also just be a property which an object either possesses or does not. Such a property is called a predicate, it is essentially a boolean valued function.
It assigns a value of True or False to each element of the domain of discourse.

For example,

| | Bird? | Flies? |
|---|---|---|
| Frog | False | False |
| Pigeon | True | True |
| Bat | False | True |
| Penguin | True | False |

Here, Bird and Flies are predicates with domain $X = \{Frog, Pigeon, Bat, Penguin\}$

A natural next step is to combine "simple" propositions to get more complex propositions.
This leads to the subject of propositional calculus.

→ Arguably, the simplest operator is the negation $\neg$ which negates the truth value of a proposition.
e.g. $\neg$Bird (Frog) is true.

→ The OR $\vee$ (also called the disjunction). $p \vee q$ is true if at least one of the propositions $p, q$ is true.
e.g. $\underbrace{\text{Bird (Frog)}}_{\text{False}} \vee \underbrace{\text{Flies (Bat)}}_{\text{True}}$ is true.

We can write the truth table of the $\vee$ as:

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{q} \\ & \vee & T & F \\ \hline p & T & T & T \\ & F & T & F \end{array} \; p \vee q$$

→ The AND $\wedge$ (also known as the conjunction). $p \wedge q$ is true if both the propositions $p, q$ are true.

e.g. Bird (Penguin) $\wedge$ Flies (Pigeon) is true.
True True

| $\wedge$ | T | F |
|---|---|---|
| T | T | F |
| F | F | F |

→ The IMPLIES $\rightarrow$, usually written using if and then. $(p \rightarrow q)$ is just equal to $(\neg p) \vee (p \wedge q) = \neg p \vee q$.

↳ If this is true, the proposition $(p \rightarrow q)$ is said to be vacuously true.

Relevant

| $\rightarrow$ | T | F |
|---|---|---|
| T | T | F |
| F | T | T |

There are several other binary operators we shall use as well; such as

$\oplus$ → XOR, $\leftrightarrow$ → IFF, $\leftarrow$ → IMPLIED BY, $\not\rightarrow$, $\not\leftarrow$, $\uparrow$ → NAND, $\downarrow$ → NOR etc.

$p \leftarrow q = q \rightarrow p$

$p \not\rightarrow q = \neg(p \rightarrow q)$

$p \not\leftarrow q = \neg(p \leftarrow q)$

$p \uparrow q = \neg(p \wedge q)$

$p \downarrow q = \neg(p \vee q)$

| $\oplus$ | T | F |
|---|---|---|
| T | F | T |
| F | T | F |

| $\leftrightarrow$ | T | F |
|---|---|---|
| T | T | F |
| F | F | T |

A single logical proposition can be expressed in several ways. All of the following are equivalent.

1. p implies q
2. if p then q
3. q if p
4. whenever p holds, q holds
5. either not p or (p and q)
6. p only if q
7. if not q then not p → this form is known as the contrapositive of "if p then q".
8. not p if not q
9. q unless not p

Also note that $p \leftrightarrow q = (p \to q) \wedge (q \to p)$

As $(p \vee q) \vee r = p \vee (q \vee r)$ and $(p \wedge q) \wedge r = p \wedge (q \wedge r)$, we can unambiguously write these as $p \vee q \vee r$ and $p \wedge q \wedge r$ respectively.

In general, we give $\neg$ the highest precedence when evaluating expressions and $\vee$ and $\wedge$ equal precedence (bracket if both are present!).

Ex. Prove that

(i) $p \to q \equiv (\neg q) \to (\neg p)$ → Contrapositive

(ii) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

(iii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

→ Associativity

(iv) $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$

(v) $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$

→ De Morgan's Laws

# Quantifiers

Suppose we want to say "Every species is a bird" (this is false), that is, "For every species $x$, Bird$(x)$ holds". This can be written as

$$\forall x \; \text{Bird}(x).$$

The universal quantifier "For all"

(In the appropriate domain)

We now want to say "Some species is a bird", that is, "There exists some $x$ such that Bird$(x)$ holds". This can be written as

$$\exists x \; \text{Bird}(x)$$

The existential quantifier "There exists"

(In the appropriate domain)

These quantifiers let us make statements about the entire domain.

Note that we can also write $\forall$ as a series of $\wedge$'s and $\exists$ as a series of $\vee$'s but this notation is significantly more compact.

$$(\forall x \; \text{Bird}(x) = \text{Bird}(\text{Frog}) \wedge \text{Bird}(\text{Pigeon}) \wedge \text{Bird}(\text{Bat}) \wedge \text{Bird}(\text{Penguin}))$$

We also have

$$\neg(\forall x \; p(x)) = \exists x \; \neg p(x).$$
$$\neg(\exists x \; p(x)) = \forall x \; \neg p(x).$$

These are just a consequence of De Morgan's Laws.

$\neg(\exists x \; p(x))$ is often denoted as $\nexists x \; p(x)$.

If there exists a unique $x$ such that $p(x)$, we write $\exists! x \; p(x)$

Consider the following predicate Likes on $X^2$, where $X = \{A, B, C\}$

Likes(p,q) denotes if p likes q

| Likes | A | B | C |
|---|---|---|---|
| A | T | T | F |
| B | F | T | T |
| C | F | F | T |

Then we have the following

$\forall x, y$ Likes$(x,y)$ Everyone likes everyone

These two are NOT the same!

$\forall x \exists y$ Likes$(x,y)$ Everyone likes someone

$\exists x \forall y$ Likes$(x,y)$ Someone likes everyone

We have the following expressions that help in the manipulation of quantifiers.

1. $\forall x \forall y\, p(x,y) \equiv \forall y \forall x\, p(x,y)$ $\quad \exists x \exists y\, p(x,y) \equiv \exists y \exists x\, p(x,y)$

Let $R$ be a proposition not involving $x$.

2. $\forall x\, (p(x) \vee R) \equiv (\forall x\, p(x)) \vee R$ $\quad \exists x\, (p(x) \wedge R) \equiv (\exists x\, p(x)) \wedge R$
3. $\forall x\, (p(x) \wedge R) \equiv (\forall x\, p(x)) \wedge R$ $\quad \exists x\, (p(x) \vee R) \equiv (\exists x\, p(x)) \vee R$
4. $\forall x\, (R \to p(x)) \equiv R \to (\forall x\, p(x))$ $\quad \exists x\, (R \to p(x)) \equiv R \to (\exists x\, p(x))$

   (Just a consequence of 2 and 3)
5. $\forall x\, (p(x) \to R) \equiv (\exists x\, p(x)) \to R$ $\quad \exists x\, (p(x) \to R) \equiv (\forall x\, p(x)) \to R$
6. $\forall x\, (p(x) \wedge q(x)) \equiv (\forall x\, p(x)) \wedge (\forall x\, q(x))$ $\quad \exists x\, (p(x) \vee q(x)) \equiv (\exists x\, p(x)) \vee (\exists x\, q(x))$
7. $(\forall x\, p(x)) \vee (\forall x\, q(x)) \equiv \forall x \forall y\, p(x) \vee q(y)$ $\quad (\exists x\, p(x)) \wedge (\exists x\, q(x)) \equiv \exists x \exists y (p(x) \wedge q(y))$

Proof of 7.

$$(\forall x\, p(x)) \vee (\forall x\, q(x)) \equiv \forall x\, (p(x) \vee \forall y\, q(y))$$
$$\equiv \forall x \forall y\, p(x) \vee q(y)$$