Graphs

Def. A simple graph is a pair of sets $G=(V,E)$ where
$E \subseteq \{\{a,b\} : a,b \in V,\ a \neq b\}$.

Recall how we used graphs for relations. (We did use directed graphs there though — basically (a,b) instead of $\{a,b\}$).

A simple graph is basically a symmetric irreflexive relation.
($\{a,b\}$ is modelled as (a,b) and (b,a))

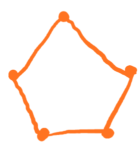
In a non-simple graph, we allow more than one edge between a pair of nodes (multigraph) or more generally, we label edges with weights.

The complete graph K_n is the graph with n nodes and all possible edges between them.
$(E = \{\{a,b\} : a,b \in V \text{ and } a \neq b\})$

K_5

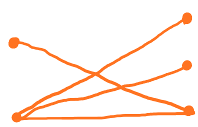
It is also known as a clique of size n.

A cycle C_n is a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and
$E = \{\{v_i, v_j\} : j = i+1 \text{ or } i=n \text{ and } j=1\}$



C_5

A graph is said to be bipartite if there exist non-empty disjoint sets V_1 and V_2 such that $V = V_1 \cup V_2$ and
$E \subseteq \{\{a,b\} : a \in V_1 \text{ and } b \in V_2\}$. (there are no edges within a part)



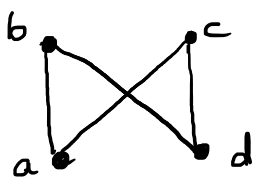
For even n, C_n is bipartite

A complete bipartite graph K_{n_1,n_2} is a bipartite graph with $|V_1| = n_1$, $|V_2| = n_2$, and $E = \{\{a,b\} : a \in V_1, b \in V_2\}$.
$(|E| = n_1 n_2)$

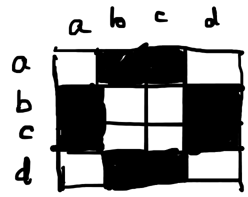
Def. Graphs $G_1 = (E_1, V_1)$ and $G_2 = (E_2, V_2)$ are isomorphic if there is a bijection $f: V_1 \to V_2$ such that $\{u,v\} \in E_1$ iff $\{f(u), f(v)\} \in E_2$.

(They have the same "structure")

We can also describe graphs by adjacency matrices:



is



Then two graphs are isomorphic if we can permute the rows and columns (in the same sense) of one matrix to get the other.

A computational problem is to determine if two graphs are isomorphic from their adjacency matrices.
There is no general efficient algorithm known for the above for large graphs.

Def. A subgraph of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$.

To get a subgraph,

1. Remove zero or more vertices along with the edges incident on them.
2. Further remove zero or more edges.

We get an induced subgraph by omitting the latter step.

Walks and Paths

Def. A walk of length $k \geq 0$ from node a to node b is a sequence of nodes $(a = v_0, v_1, \dots, v_k = b)$ such that for all $0 \leq i \leq k-1$, $\{v_i, v_{i+1}\} \in E$.

If a walk has no repeating nodes, it is a path.

If a walk of length $k \geq 3$ has $v_0 = v_k$ and has no other repeating nodes, it is called a cycle.
A graph is acyclic if it has no cycles (there is no subgraph isomorphic to C_k)

Def. Nodes u and v are said to be connected if there exists a path from u to v.
Equivalently, they are connected if there is a walk between them. (Why?)

The connectedness relation is an equivalence relation.
The equivalence classes of this relation are called the connected components of G.

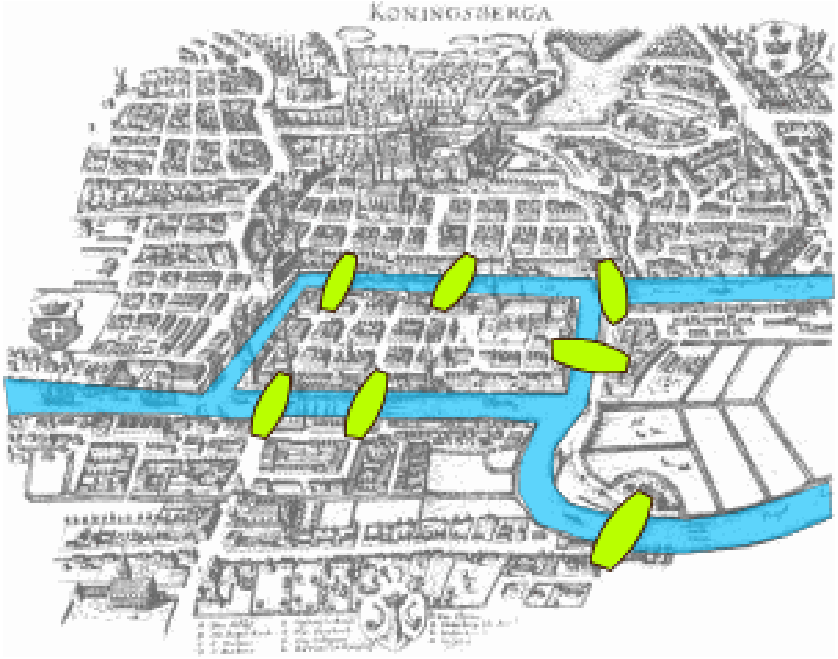
Given a simple graph $G = (V, E)$, the degree of $v \in V$ is the number of edges incident on v. That is,

$$\deg(v) = |\{u : \{u, v\} \in E\}|$$

Note that $2|E| = \sum_{v \in V} \deg(v)$ (each edge is counted twice)

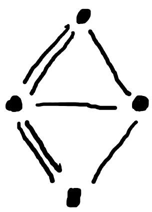
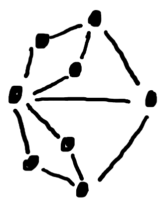
The degree sequence of a graph is a sorted list of degrees.
It is invariant under isomorphism.

We write $\Delta(G) = \max_{v \in V} \deg(v)$.



A famous question known as the "Seven Bridges of Königsberg" asks whether it is possible to walk through the city crossing each bridge exactly once.

We can model this as a graph:

 or equivalently, as a simple graph: 

With this motivation, define an Eulerian trail as a walk visiting each edge exactly once.

The question then asks if an Eulerian trail exists for the latter graph.

Note that if an Eulerian trail exists, there must be at most 2 odd degree nodes.

Indeed, define

$$\text{Enter}(v) = \{\{v_{i-1}, v_i\} : v_i = v\}$$

$$\text{Exit}(v) = \{\{v_i, v_{i+1}\} : v_i = v\}$$

that partitions all edges incident on v. Further, $|\text{Enter}(v)| = |\text{Exit}(v)|$ for all v except the start and end nodes.

$\Rightarrow$ There can be at most two odd degree nodes.

(Namely, the start and end)

As the graph drawn above has 3 odd degree nodes no Eulerian walk exists.

An Eulerian circuit is a closed walk ($v_0 = v_k$) visiting each edge exactly once.
If an Eulerian circuit exists, there are no odd degree nodes.

Further, if there are no odd degree nodes and all edges appear in a single connected component, there must exist an Eulerian circuit!
(Try splitting it into several cycles and "stitching" them together)
This also gives an efficient algorithm to find an Eulerian circuit if it exists.

A Hamiltonian cycle is a cycle that contains all nodes in the graph.

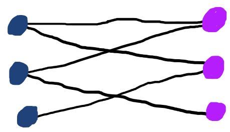
There is no efficient algorithm known to check if a graph has a Hamiltonian cycle.
Indeed, this is an "NP-hard" problem.

Given connected nodes u and v, the distance between them is the length of a shortest walk between them. (and ∞ if no walk exists)
this shortest walk must be a path.

We also define the diameter of a graph as the largest distance between two nodes in a graph.
(This is ∞ if the graph is not connected)

Graph Coloring

We can "color" bipartite graphs using two colors such that there is no edge between two nodes of the same color.



More generally, a coloring using k colors is proper if there is no edge between nodes of the same color.
A function $c: V \to [k]$ such that
$$\forall x, y \in V \quad \{x,y\} \in E \to c(x) \neq c(y)$$
The least number of colours possible in a proper colouring of G is called the chromatic number of G, denoted $\chi(G)$.

Suppose H is a subgraph of G. Then
G has a k-coloring $\Rightarrow$ H has a k-coloring.
That is, $\chi(H) \leq \chi(G)$.
In particular, if K_n is a subgraph of G, $\chi(G) \geq n$
If C_n for odd n is a subgraph, $\chi(G) > 2$.

Ex. Prove that isomorphic graphs have equal chromatic number.

There is no efficient algorithm known to calculate chromatic number.
(It is NP-hard)

Let us consider bipartite graphs.

For all integer $n \geq 1$, C_{2n+1} is not bipartite.
(Left as an exercise, try induction)

This extends to the following theorem.

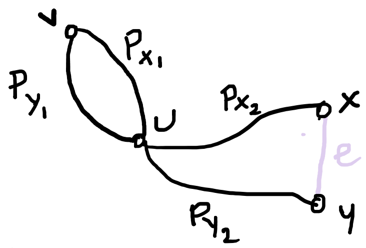
Theo. A graph G (with $|V|>1$) is bipartite iff it has no odd cycle.

Proof. Let G be not bipartite.
Then G must contain a connected component that is bipartite
(with >1 nodes) (why?)

Fix some v in this component and let

$$A = \{x : \text{dist}(x,v) \text{ is even}\}$$
$$B = \{x : \text{dist}(x,v) \text{ is odd}\}$$

As it is not bipartite, there exists edge $e=\{x,y\}$ where $x,y \in A$ or $x,y \in B$.
Let P_x and P_y be the shortest paths from v to x and v to y.
Suppose P_x and P_y both pass through node $u \neq v$.



Observe that $|P_{x_1}| = |P_{y_1}|$ due to the minimality of $|P_x|$ and $|P_y|$.

As $|P_x|$ and $|P_y|$ are either both even or both odd, $|P_{x_2}|$ and $|P_{y_2}|$ are either both even or both odd as well.
Using this, we can assume that P_x and P_y intersect only at v. (Otherwise, we can do this reduction repeatedly)
Then $P_x \cup P_y \cup \{e\}$ is a cycle of odd length!
(P_x, then e, then P_y^{rev})

For the converse. when G has no odd cycle, the coloring corresponding to

$$A = \{x : \text{dist}(x,v) \text{ is even}\}$$
$$B = \{x : \text{dist}(x,v) \text{ is odd}\}$$

is a valid one.

Let G have n nodes. $\chi(G) = n \longleftrightarrow G$ is isomorphic to K_n.

(If G is not isomorphic to K_n, there exist u, v s.t. $\{u, v\} \notin E$
Color u and v in the same color and every other vertex
differently $\Rightarrow \chi(G) \leq n-1$)

For a graph G, the clique number $\omega(G)$ is the largest k such that G has a subgraph isomorphic to K_k.

$$\chi(G) \geq \omega(G)$$

The independence number $\alpha(G)$ is the largest k such that G has a set of k nodes with no edges among them.

(Such a subgraph with no edges is an anticlique)

In a colouring, the nodes of each colour must correspond to an anticlique.

Consider a coloring of G with $\chi(G)$ nodes.

$$n = \sum_c |\text{nodes with colour } c| \leq \chi(G)\,\alpha(G)$$

$$\chi(G) \geq \frac{n}{\alpha(G)}$$

Theo. For any G, $\chi(G) \leq \Delta(G) + 1$

Proof Try induction on the number of nodes.

Suppose for all $G = (V, E)$ with $|V| = k$, $\chi(G) \leq \Delta(G) + 1$

Let $G = (V, E)$ with $|V| = k+1$.

Let $G' = (V', E')$ be obtained from G by removing some $v \in V$.
(and all incident edges)

$$\chi(G') \leq \Delta(G') + 1 \leq \Delta(G) + 1.$$

Colour G' with $\Delta(G)+1$ colours.

However, $\deg(v) \leq \Delta(G)$. Just colour v with a colour in $\{1, \dots, \Delta(G)+1\}$ that does not appear in its neighbourhood.

$\Rightarrow \chi(G) \leq \Delta(G) + 1$

Note that this describes a (recursive) algorithm to colour a graph G in $\Delta(G)+1$ colours.

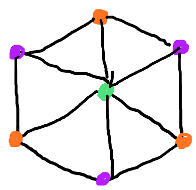
Note that equality holds in the above in K_n and C_{2n+1}.
Turns out that these are the only cases where equality holds.

A graph with no cycles is known as a forest.

The path graph P_n has
$V = \{1, 2, \cdots, n\}$ and $E = \{\{i, i+1\} : i \in [n-1]\}$

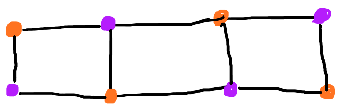
(P_7)

The wheel graph W_n $(n \geq 3)$ has
$V = \{hub\} \cup \mathbb{Z}_n$ (hub is some specific node)
$E = \{\{x, hub\} : x \in \mathbb{Z}_n\} \cup \{\{x, x+1\} : x \in \mathbb{Z}_n\}$



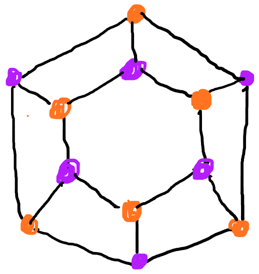
(W_6) Note that C_n is a subgraph of W_n.

The ladder graph L_n has
$V = \{0,1\} \times [n]$
$E = \{\{(0,i), (1,i)\} : i \in [n]\} \cup \{\{(b,i), (b,i+1)\} : b \in \{0,1\}, i \in [n-1]\}$



(L_4) Note that P_n is a subgraph of L_n (up to isomorphism)

The circular ladder graph CL_n is the same as L_n but with two additional edges: $\{(b,n), (b,1)\}$ for $b \in \{0,1\}$.



(CL_6)

The hypercube graph Q_n has

V = set of all n-bit strings.

$E = \{\{x,y\} : x \text{ and } y \text{ differ at exactly one position}\}$

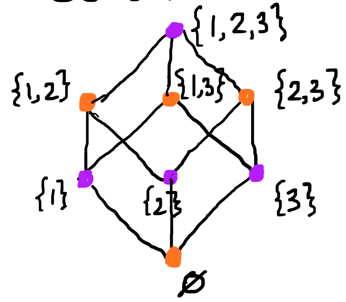
Q_n has 2^n nodes but the diameter $\left(\max_{x,y \in V} \text{dist}(x,y)\right)$ is only n.

It is n-regular and bipartite.

↳ label nodes with even no. of 1s in one colour.
odd no. of 1s in another.

Q_{n-1} is a subgraph of Q_n (up to isomorphism)

Now consider the graph isomorphic to Q_n where nodes correspond to subsets of $[n]$ and edges exist between subsets that differ by a single element.



Consider the graph $KG_{n,k}$ which has the nodes at the k^{th} level of this, namely the subsets of size k. Insert an edge between any two nodes that <u>not</u> intersect.

A clique in $\overline{KG_{n,k}}$ is a set of subsets which intersect pairwise.

For example, $\{\{n\} \cup S : S \subseteq [n-1], |S| = k-1\}$ is a clique with $C(n-1, k-1)$ nodes.

The Erdös-Ko-Rado Theorem states that if $k \le n/2$, then there are no larger cliques than this.

The Kneser Graph $KG_{n,k}$ is the complement of this graph, that is, there is an edge between disjoint subsets.

So in $KG_{n,k}$, if $k \le n/2$, the largest anticlique is of size $C(n-1, k-1)$.

Let us define the "complement" we used above more concretely. The complement of $G=(V,E)$ is $\overline{G} = (V, \overline{E})$, where $\overline{E} = \underbrace{\{\{a,b\} : a,b \in V,\ a \neq b\}}_{\text{this set is sometimes denoted } C(V,2)} \setminus E$.

We can similarly define the union, intersection, difference and symmetric difference of two graphs with the same vertex set.

We can further extend this to the union/intersection of $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ by taking the union intersection of the vertex sets and edge sets separately.

Given $G=(V,E)$, $G^2=(V,E')$ where

$$E' = E \cup \{\{x,y\} : \exists w\ \{x,w\},\{w,y\} \in E\}$$

More generally, G^k has an edge $\{x,y\}$ iff $\mathrm{dist}(x,y) \le k$.

Given $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, their cross product or tensor product is $G_1 \times G_2 = (V_1 \times V_2, E)$ where

$$\{(u_1,v_1),(u_2,v_2)\} \in E \text{ iff } \{u_1,u_2\} \in E_1 \text{ and } \{v_1,v_2\} \in E_2.$$

For any graph G, $G \times K_2$ is a bipartite graph.

Given $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, their box product is $G_1 \Box G_2 = (V_1 \times V_2, E)$ where

$$\{(u_1,v_1),(u_2,v_2)\} \in E \text{ iff } (\{u_1,u_2\} \in E_1 \text{ and } v_1=v_2) \text{ or } (\{v_1,v_2\} \in E_2 \text{ and } u_1=u_2)$$

For example, $Q_m \square Q_n = Q_{m+n}$ (isomorphically).

The Hamming graph $H_{n,q}$ is

$$K_q \square K_q \square \cdots \square K_q \quad (n \text{ copies})$$

Vertex set is $[q]^n$. There is an edge between two nodes if they are the same for all but one co-ordinate.

Matchings

Def. A matching in a graph $G=(V,E)$ is a set of edges which do not share any vertex.

A set $M \subseteq E$ s.t. for all $e_1, e_2 \in M$, $e_1 \neq e_2 \Rightarrow e_1 \cap e_2 = \emptyset$.

$\emptyset$ and $\{e\}$ are trivial matchings.
A matching M is perfect if all nodes are matched by M.
$(\forall v \in V, \ \exists e \in M \text{ s.t. } v \in E)$

An algorithmic task is to find the largest matching in a graph.

What about matchings in bipartite graphs?
Denote by $G=(X,Y,E)$ a bipartite graph $(X \cup Y, E)$ where $X, Y \neq \emptyset$, $X \cap Y = \emptyset$, and for all $e \in E$, $|e \cap X| = |e \cap Y| = 1$.
Then a complete matching from X to Y is a matching M such that $|M| = |X|$.
Such a matching exists only if $|X| \leq |Y|$.
If $|X| = |Y|$, then a complete matching from X to Y is a complete matching from Y to X, and thus a perfect matching.

Def. Given a graph $G=(V,E)$ and $v \in V$, the neighbourhood of v is

$$\Gamma(\{v\}) = \{u : \{u,v\} \in E\}.$$

For any $S \subseteq V$,

$$\Gamma(S) = \bigcup_{v \in S} \Gamma(\{v\})$$

In a bipartite graph $G=(X,Y,E)$, if $S \subseteq X$, then $\Gamma(S) \subseteq Y$.
We say that S is shrinking if $|\Gamma(S)| < |S|$.
For $B \subseteq Y$, we say that S is shrinking in B if $|\Gamma(S) \cap B| < |S|$.

Theo. [Hall's Theorem]

A bipartite graph $G=(X,Y,E)$ has a complete matching from X to Y if and only if no subset of X is shrinking.

Proof. If there is a complete matching from X to Y, then $\forall S \subseteq X$, there must be a distinct element in Y corresponding to each x (corresponding to the matching) $\Rightarrow |\Gamma(S)| \geq |S|$

Now suppose every subset of X is shrinking.
We perform strong induction on $|X|$.
If $|X|=1$, there is trivially a complete matching.
Suppose it holds for $|X| \leq k$.
Now let $|X|=k+1$ s.t. for all $U \subseteq X$, $|\Gamma(U)| \geq |U|$.
Pick some arbitrary $x \in X$ and neighbour y of x.
(such a y exists as $\{x\}$ is not-shrinking)

CASE I. If there is a complete matching from $X \setminus \{x\}$ to $Y \setminus \{y\}$, then there is a complete matching from X to Y.

CASE II. Suppose there is no complete matching from $X \setminus \{x\}$ to $Y \setminus \{y\}$.
By the induction hypothesis, there is some $S \subseteq X \setminus \{x\}$ s.t. S is shrinking in $Y \setminus \{y\}$.
However, S is shrinking in Y. This implies that $|\Gamma(S)| = |S|$.
$(y \in \Gamma(S))$

As $|S| \leq k$ and no subset of S is shrinking, there must be a complete matching of S into $\Gamma(S)$.
Further, we claim that there is a complete matching from $X \setminus S$ into $Y \setminus \Gamma(S)$.
As $|X \setminus S| \leq k$, it is enough to show that for all $T \subseteq X \setminus S$, $|\Gamma(T) \setminus \Gamma(S)| \geq |T|$
Then use the fact that $|\Gamma(T \cup S)| \geq |T| + |S|$

$$\Rightarrow |\Gamma(T \cup S)| \geq |T| + |\Gamma(S)|$$
$$\Rightarrow |\Gamma(T \cup S) \setminus \Gamma(S)| \geq |T|$$
$$\Rightarrow |\Gamma(T) \setminus \Gamma(S)| \geq |T|$$

$\Rightarrow$ There is a complete matching from $X \setminus S$ into $Y \setminus \Gamma(S)$ and hence from X into Y.

Corollary. The edge set of a regular bipartite graph can be partitioned into d matchings, where d is the degree of any vertex.
Note that if $G=(X,Y,E)$ satisfies the above, $|X|=|Y|=\frac{|E|}{d}$

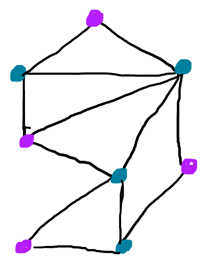
Proof. We prove this using induction on d.
If $d=1$, the graph is a matching.
Suppose it holds for $d=k$.
Let $G=(X,Y,E)$ be of degree $d=k+1$.
For any $S\subseteq X$,

$$\underbrace{d|S|}_{\substack{\text{no. of edges}\\ \text{incident on } S}} \leq \underbrace{d\,|\Gamma(S)|}_{\substack{\text{no. of edges}\\ \text{incident on } \Gamma(S)}} \quad\rightarrow\quad |S|\leq \Gamma(S)$$

$\Rightarrow$ There is a perfect matching in G. (by Hall's Theorem)
Removing this matching, all nodes in the remaining graph have degree d. Using the inductive hypothesis proves the result.

Vertex Covers

Def. A vertex cover of a graph $G=(V,E)$ is a set C of vertices such that every edge is incident on at least one vertex of C.
($\forall e\in E,\ e\cap C\neq\emptyset$)
Sort of like bipartite but only one side.



The set of blue vertices is a vertex cover of the entire graph.

For any $v\in V$, V and $V\setminus\{v\}$ are (trivial) vertex covers.
A hard problem is to determine the size of a smallest vertex cover.
(NP-hard, in fact)

In bipartite graphs, the size of a smallest vertex cover is equal to that of a largest matching.
In general graphs, they are within a factor of 2 of each other.

Theo. Let $G=(V,E)$ be a graph, C be a vertex cover of G, and M be a matching in G. Then $|C| \geq |M|$.

Proof. Indeed, any vertex in C can cover at most one edge in M.

Theo [König's Theorem]
In a bipartite graph, the size of the smallest vertex cover is the size of the largest matching.

Proof. Let C be a smallest vertex cover and the graph be $G=(X,Y,E)$.
Let $A = C \cap X$ and $B = C \cap Y$.
We shall show that there is a complete matching from A to $Y \setminus B$.
For any $S \subseteq A$, suppose that S is shrinking. Then $C \cup T(S) \setminus S$ is a vertex cover as well, and it has cardinality less than that of C, which is a contradiction. By Hall's Theorem, there is a complete matching from A to $Y \setminus B$ and similarly, a complete matching from B to $Y \setminus A$.
This implies the existence of a matching M such that $|M| \geq |C|$, and the result follows.

The edges covered by S are covered by $T(S)$.

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While finding a minimum vertex cover is a difficult task, finding a maximum matching isn't.

Def. A matching M in a graph $G=(V,E)$ is a maximal matching if there exists no $e \in E \setminus M$ such that $M \cup \{e\}$ is a matching.

We can easily construct maximal matchings by repeatedly choosing an arbitrary edge and deleting all edges touching it until there are no remaining edges.

Theo. Let M be a maximal matching in a graph G. There is then a vertex cover C of G of size $2|M|$.

Proof. Consider the set $C = \bigcup_{e \in M} e$. If C was not a vertex cover, there would exist an edge e such that $e \cap C = \emptyset$. However, as M is a maximal matching, no such edge exists and C must be a vertex cover.

Corollary. If C is a smallest vertex cover and M is a maximal matching, $|M| \leq |C| \leq 2|M|$.

Finding the smallest maximal matching is NP-hard as well.

Def. In a graph $G = (V, E)$ with $I \subseteq V$, I is called an independent set or anticlique if no two vertices in I are adjacent.

$(\forall e \in E,\ e \not\subseteq I)$

Theo. I is an anticlique if and only if $\overline{I}$ is a vertex cover.

Proof. I is an anticlique $\Leftrightarrow \forall e \in E \ e \not\subseteq I$

$\Leftrightarrow \forall e \in E \ e \cap \overline{I} \neq \emptyset$

$\Leftrightarrow \overline{I}$ is a vertex cover

Finding the smallest vertex cover is equivalent to finding the largest anticlique.

Trees

Def. A forest is an acyclic graph.
A tree is a connected acyclic graph.

Each connected component of a forest is a tree.
Any subgraph of a forest is a forest.
(cycles cannot be created)

Def. Given a tree, a leaf is a node of degree 1.

Theo. Any tree with at least 2 nodes has at least 2 leaves.

Proof. Consider a maximal path $v_0 \cdots v_k$ (there is no v_{k+1} s.t. $v_0 \cdots v_k v_{k+1}$ is a path)
We must have $k>0$ (as it is a tree).
If v_0 is not a leaf, it must have a neighbour v_i (for $1 \leq i \leq k$). However, then, $v_0 \cdots v_i$ form a cycle! Thus, v_0 (and similarly, v_k) must be a leaf.

If T is a tree (with ≥ 2 nodes), deleting a leaf (with the corresponding edge) results in a tree.

Theo. Given a tree G and nodes u, v, there is exactly one path from u to v.
Proof. We perform induction on the number of nodes.
If there is only one node, then there is clearly exactly one path from the node to itself.
Suppose the hypothesis holds for k nodes. Now, if G has $k+1$ nodes, delete a leaf ℓ to get a tree G' (of k nodes).
By the induction hypothesis, it only remains to check that there is a unique u-ℓ path (for $u \neq \ell$). To see this, let v such that $\{v, \ell\}$ is an edge. As there is a unique u-v path and any u-ℓ path must go through v, the result follows.

Theo. Let $G=(V,E)$ be a tree. Then $|E|=|V|-1$.
Proof. Perform induction on $|V|$. If $|V|=1$, the result clearly holds. Otherwise, delete a leaf and use the induction hypothesis to obtain the result.

Theo. If a graph $G=(V,E)$ is connected and $|E|=|V|-1$, it must be a tree.

Proof. If there is a cycle, then we can delete any edge in the cycle and the graph will still be connected.
If we repeat this until there are no cycles remaining, we obtain a tree $T=(V,E')$ such that $|E'|<|V|-1$. This proves the result.

Theo. Let $G=(V,E)$ be a forest. Then the number of connected components is $|V|-|E|$.

This follows from the fact that if $G_i=(V_i,E_i)$ is a connected component, then $|V_i|-|E_i|=1$.

The above implies that deleting a degree d node from a tree results in a forest with $d-1$ connected components.

Dilworth's Theorem

Recall Mirsky's Theorem:

The least number of anti-chains needed to partition S is exactly the size of a largest chain.

size of any chain $\leq$ size of any anti-chain decomposition
(The theorem says that equality can be attained)

This is similar (in structure) to the result

size of any matching $\leq$ size of any vertex cover
(König's theorem says that equality can be attained in a bipartite graph)

The dual of Mirsky's Theorem is Dilworth's Theorem:

Size of any antichain $\leq$ size of any chain decomposition.
(The theorem says that equality can be attained)

Theo. [Dilworth's Theorem]

The least number of anti-chains needed to partition a poset S is exactly equal to the size of a largest chain.

Proof. It is easily shown that

$$\text{size of any anti-chain} \leq \text{size of any chain decomposition}$$

Now, let $|S| = n$.

We shall construct a bipartite graph G such that

- if there is a vertex cover of size $\leq t$ in G, there is an anti-chain of size $\geq n-t$
- if there is a matching of size $\geq t$ in G, there is a chain decomposition of size $\leq n-t$.

Then König's theorem implies the result.

Let $G = (S\times\{0\}, S\times\{1\}, E)$, where $E = \{\{(u,0),(v,1)\} : u \leq v,\ u \neq v\}$

- If C is a vertex cover of size t in G, then let

  $$A = S \setminus \{p \in S : (u,0) \in C \text{ or } (u,1) \in C\}$$

  Then for any $x, y \in A$ $(x \neq y)$, we cannot have $x \leq y$.
  (as then we must have $(x,0)$ and $(y,1) \in C$)

  As $|A| \geq |S| - |C| = n - |C|$, the first part follows.

- If M is a matching of size t in G, then consider the graph $F = (S, E')$ where $E' = \{\{u,v\} : \{(u,0),(v,1)\} \in M\}$.

  F is a forest where each connected component is a path:
  - → Any $u \in F$ has degree ≤ 2 (the edges incident on $(u,0)$ and $(u,1)$)
  - → F is acyclic (if $v_0 v_1 \cdots v_k$ is a cycle, $v_0 \leq v_1 \leq \cdots \leq v_k \leq v_0$ which is not possible)

  Each of these paths form a chain in the poset and the number of connected components is $|S| - |E'| = n - t$.
  Therefore F is essentially a chain decomposition of $n-t$ parts.

Using König's Theorem, this completes the proof.

Mirsky's Theorem and Dilworth's Theorem are statements about the comparison graph of the poset.

If $(S, \leq)$ is a poset, its comparison graph is $G = (S, E)$ where $E = \{\{u, v\} : u \leq v \text{ and } u \neq v\}$

If G is a comparison graph, any subgraph of G is a comparison graph as well.

With this in mind, if S is a poset and G is its comparison graph.

→ a chain in S is just a clique in G while an anti-chain is an anticlique.

→ a chain decomposition of S is a colouring of G while an anti-chain decomposition is a colouring of $\bar{G}$.

[Mirsky's and Dilworth's Theorems]

If G is a comparison graph, $\omega(G) = \chi(G)$.

If G is a comparison graph, $\omega(\bar{G}) = \chi(\bar{G})$.

Def. A graph G is a perfect graph if for every induced subgraph of G, $\chi(G) = \omega(G)$.

As it turns out, there is a (non-trivial) result that states that a graph is perfect if and only if its complement is perfect.

(Perfect Graph Theorem)

Given the perfect graph theorem, either of Mirsky's and Dilworth's Theorems can easily be proved from the other.

Any comparison graph is perfect.

The complement of any comparison graph is perfect.