Counting

Permutations and Combinations

Def Given a non-empty finite set B (known as an alphabet), a string of length $k \in \mathbb{N}$ in B is a mapping $\sigma : \{1, \dots, k\} \to B$.

There exist n^k length-k strings over an alphabet of size n.
This can easily be proved by induction on k.
Note that the empty string is well-defined.
A binary string is a string over an alphabet of size 2.
A length-k binary string can be used to represent a subset of a set of size k.
More concretely, let $[k] = \{1, 2, \dots, k\}$ and $B = \{0, 1\}$. Then given any σ, we can get a subset of $[k]$ by $\{i : \sigma(i) = 1\}$.
Further note that there is a one-one mapping between subsets and strings.
We often represent $\sigma(i)$ as σ_i.

A permutation of B, an alphabet of size n, is a bijection from $[n]$ to B.
This is just a length n string with no repeating characters.
For example, $\underset{1}{c}\,\underset{2}{a}\,\underset{3}{d}\,\underset{4}{e}\,\underset{5}{b}$ is a permutation of $\{a, b, c, d, e\}$.

If we loosen the criteria by considering any string with no repeating characters, we have a one-one function from $[k]$ to B instead of a bijection.

Let $P(n,k)$ represent the number of such length k strings with no repeating characters given an alphabet of size n.

If $k > n$, then $P(n,k) = 0$. Otherwise, we claim $P(n,k) = \frac{n!}{(n-k)!}$.
(Here, $n! = \begin{cases} 0, & n = 0 \\ n(n-1)!, & n > 0 \end{cases}$) This is easily proved by induction on n.
$(P(n,k) = nP(n-1, k-1))$

On the other hand, how many subsets of size k does a set of size n have?

We can represent subsets as strings without repetition. ($\{a,b,c\} = \underset{1\,2\,3}{abc}$)

Here, the same subset can be represented by multiple strings.

(abc and bca represent the same set)

We know how many such strings there are! Exactly $k!$ strings which involve those k symbols.

Therefore, the number of subsets is size k is $\dfrac{P(n,k)}{k!} = \dfrac{n!}{k!(n-k)!}$

We represent this as $C(n,k)$ or $\binom{n}{k}$ (read "n choose k")

Note that $C(n,k) = C(n, n-k)$.

(choosing a subset of size k is the same as choosing a complement of size $n-k$)

$C(0,0) = 1$. This just says $\emptyset \subseteq \emptyset$.

$C(n,0) + C(n,1) + \cdots + C(n,n) = 2^n$ (there are 2^n subsets)

More generally,

Theo. [Binomial Theorem]

For any $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$,

$$(1+x)^n = \sum_{k=0}^{n} C(n,k)\, x^k$$

think of this as choosing which k x's to multiply.

This can be proved using induction. To do so, show that

$$C(n,k) = C(n-1,k-1) + C(n-1,k)$$

This can be thought of as: let $|S| = n$ and $a \in S$. Count the number of subsets of size k that include a and the number that don't include a.

$C(n-1,k-1)$

$C(n-1,k)$

We now consider a series of "balls and boxes" problems.

In how many ways can you throw a set of balls into a set of boxes?

This has different answers depending on whether the balls/boxes are (in)distinguishable.

Boxes \ Balls	Labelled	Unlabelled
Labelled	Function	Multiset
Unlabelled	Set Partition	Integer Partition

We can also have some more variants: no box is empty

atmost one ball per box.

Distinguishable balls/Distinguishable Boxes

This is just a function from A, the set of balls to B, the set of bins.

(each ball is thrown into a single bin)

The number of ways of throwing is just the number of functions from A to B. Such a function can be represented a string of length $|A|$ over B.

$\Rightarrow$ The number of functions from A to B is $|B|^{|A|}$.

If every box can hold atmost one ball, the function is one-one, which corresponds to a permutation of length $|A|$ over B.

$\Rightarrow$ The number of one-one functions from A to B is $P(|B|, |A|)$

If no box is empty, the function is onto.

Recall the inclusion-exclusion principle: $|S \cup T| = |S| + |T| - |S \cap T|$.

More generally, given finite sets $T_1, \ldots, T_n$,

$$\left| \bigcup_{i \in [n]} T_i \right| = \sum_{\substack{J \subseteq [n] \\ J \neq \emptyset}} (-1)^{|J|+1} \left| \bigcap_{j \in J} T_j \right|$$

Prove this by induction on n.

Without loss of generality, let $A = [k]$ and $B = [n]$. We denote the number of onto functions from A to B by $N(k,n)$

We claim

$$N(k,n) = \sum_{i=0}^{n} (-1)^i \, C(n,i) \, (n-i)^k$$

$$\hookrightarrow n^k - C(n,1)(n-1)^k + \cdots$$

The set of non-onto functions is

$$\left(\bigcup_{i \in [n]} T_i \right) \text{ where } T_i = \{ f : A \to B \mid i \notin \mathrm{Im}(f) \}$$

Use the inclusion-exclusion principle! The cardinality of this set is

$$\sum_{\substack{J \subseteq [n] \\ J \neq \emptyset}} (-1)^{|J|+1} \left| \bigcap_{j \in J} T_j \right| = \sum_{\substack{J \subseteq [n] \\ J \neq \emptyset}} (-1)^{|J|+1} (n - |J|)^k$$

f is just a function from $[k]$ to $[n] \setminus J$

$\Rightarrow$ there are $(n-|J|)^k$ such functions

Given some $|J|$, how many such J exist? This number is equal to $C(n,|J|)$. Letting $i = |J|$ on the right,

$$\Rightarrow \left| \bigcup_{i \in [n]} T_i \right| = \sum_{j=1}^{n} (-1)^{j+1} C(n,j) \, (n-j)^k$$

Therefore, the number of onto functions is n^k – this, namely

$$\boxed{N(k,n) = \sum_{i=0}^{n} (-1)^i \, C(n,i) \, (n-i)^k}$$

Indistinguishable balls / Distinguishable boxes

We only care about the number of balls in each box, not the identity of the balls.

A **multi-set** is like a set, but allows elements to occur multiple times. Only multiplicity matters: $[a,a,b] = [a,b,a]$.

A multi-set is just a multiplicity function $\mu : B \to \mathbb{N}_0$.

The size of a multi-set is the sum of its multiplicities.

Here, the question is equivalent to finding the number of multisets of size k with the base set as $[n]$.

We want the number of $(n_1, n_2, \cdots, n_n)$ with

$$n_1 + n_2 + \cdots + n_n = k \qquad (\text{Here } n_n = \mu(n))$$

$\hookrightarrow$ multiplicity

Any such thing can be represented with $n-1$ "bars" and k "stars".

So $(n_1, n_2, n_3, n_4, n_5) = (1, 2, 1, 0, 1)$ is

$$* \mid * * \mid * \mid\mid * \qquad k=5,\ n=5$$

box 1 box 2 box 3 ...

There is a one-one correspondence between such "star-bar" combinations and the required result.

Any such combination can be formed by choosing some $n-1$ positions among to $n+k-1$ positions, filling bars there, and filling stars everywhere else.

$\Rightarrow$ The number of combinations is $C(n+k-1, n-1)$

If we want each box to be non-empty, then just throw one ball into each box and solve the problem for a smaller n $(= n-k)$

(we get $C(n-1, n-k-1)$)

If at-most one ball per box, then we just want a set of size k, so there are $C(n,k)$ possibilities.

Distinguishable balls / Indistinguishable boxes

We partition the set A of balls into unlabelled bins.

We must just find the number of partitions of A.

$\hookrightarrow$ we defined this when studying equivalence relations

How many partitions does a set A of k elements have?

Let $S(k,n)$ denote the number of partitions of $[k]$ into exactly n parts.

(This is the no bin empty case)

"Stirling number of the second kind"

More generally, the number of ways A can be partitioned into at most n parts is $\sum_{m \in [n]} S(k,m)$

Further, $B_k = \sum_{m \in [k]} S(k,m)$ is the total number of partitions of $[k]$.

Bell number.

Now, in $S(k,n)$, suppose the parts are labelled $1, 2, \cdots, n$. There are $N(k,n)$ such partitions. However, disregarding the labelling, each partition is counted $n!$ times.

$$\Rightarrow S(k,n) = \frac{N(k,n)}{n!}$$

Indistinguishable balls / Indistinguishable boxes

This problem is equivalent to writing k as a sum of n non-negative integers.

$\Rightarrow$ The number of $(x_1, x_2, \cdots, x_n)$ such that

$$x_1 + x_2 + \cdots + x_n = k \quad \text{and} \quad 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$$

If no box is empty, then each x_i is positive.

The number of such solutions is called the partition number $p_n(k)$.

If there is no restriction, the answer is just $p_n(n+k)$.

(Let $y_i = x_i + 1$ for each i)

Let us now examine the partition number.

$$p_n(k) = \left| \{ (x_1, \cdots, x_n) : x_1 + \cdots + x_n = k \text{ and } 1 \leq x_1 \leq \cdots \leq x_n \} \right|$$

First of all, $p_0(0) = 1$
$p_0(k) = 0$ for $k > 0$
$p_n(k) = 0$ if $k < n$

We then have

$$p_n(k) = \underbrace{p_n(k-n)}_{\text{the case } x_1 > 1} + \underbrace{p_{n-1}(k-1)}_{\text{the case } x_1 = 1}$$

This enables us to (recursively) define $p_n(k)$ for every k and n.

Boxes \ Balls	Distinguishable	Indistinguishable
Distinguishable	n^k (if one-one, then $P(n,k)$; if onto, then $N(k,n)$)	$C(n+k-1,k)$ (if one-one, then $C(n,k)$; if onto, then $C(k-1,n-1)$)
Indistinguishable	$\sum_{m\in[n]} S(k,m)$ (if one-one, then 0 or 1; if onto, then $S(k,n)$)	$p_n(n+k)$ (if one-one, then 0 or 1; if onto, then $p_n(k)$)

where $N(k,n) = \sum_{i=0}^{n} (-1)^i C(n,i)(n-i)^k$.

$S(k,n) = \dfrac{N(k,n)}{n!}$

$p_n(k) = p_n(k-n) + p_{n-1}(k-1)$, $p_0(0) = 1$, $p_0(k) = 0$ for $k > 0$
$p_n(k) = 0$ if $k < n$